Tandem Packet-Radio Queueing Systems

MOSHE SIDI

Abstract—**Tandem packet-radio systems are considered. The nodes of the tandem have infinite buffers, they share a common radio channel for transmisison of data packets and each node transmits whenever it has a packet ready for transmission.**

An approximate analysis is developed for an arbitrary tandem. The approximation uses the exact results obtained for a four-node tandem and is based on exploiting the special features of tandem systems. The approximation is compared with simulation results and very good agreement is observed.

I. Introduction

Tandem packet-radio systems have received some attention in the literature [1]. A schematic figure of such a system is depicted in Fig. 1. The tandem consists of N nodes having *infinite buffers* that transmit data over a common shared radio channel. Fixed-length data packets enter the system at the nodes from corresponding sources. It is assumed that time is divided into slots of size corresponding to the transmission time of a packet and transmissions are started only at the beginning of a slot. The nodes of the tandem have radio transmitters with omnidirectional antennas and their transmission range is such that a transmission at node $i(2 \leq i \leq N-1)$ can be heard only by nodes $(i-1)$ and $(i+1)$. Nodes 1 and N can be heard at nodes 2 and $N-1$, respectively. The final destination of all packets is a station (Fig. 1) that receives the packets transmitted by node 1, and packets entering the system at any node i are forwarded via nodes $i-1, i-2, \cdots, 1$ till they finally reach the station. We assume that a node can transmit only one packet in a given slot. Similar tandem models but for point-to-point systems have been studied in [6]–[8].

In this paper we consider the statistical behavior analysis of tandem systems in which each node transmits the packet at the head of its queue, whenever its queue is nonempty. The transmission of a node will be successful only if the two nodes ahead of it in the tandem will be silent, i.e., empty. A packet whose transmission is not successful (the feedback upon success or failure is assumed to be instantaneous) remains at the head of the queue at the respective node.

We start with a tandem where packets arrive only at node N (the "top" node). Then we focus on a tandem with four nodes where packets arrive to all four nodes from their corresponding sources. The motivation for considering a four-node tandem is twofold. Firstly, it is the simplest tandem where channel reuse can take place (nodes 1 and 4 can both transmit successfully if nodes 2 and 3 are silent). Secondly, it will serve us as a crucial building stone in developing our approximate analysis of a general tandem. Finally, we introduce the approximate analysis of an arbitrary tandem. The approximation uses the exact results obtained for the four-node tandem and is based on exploiting the special features of tandem systems. The approximation is presented for independent

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The author is with the Department of Electrical Engineering, Technion-Israel Institute of Technology, Haifa 32000, Israel. He is currently on leave at the IBM Thomas J. Watson Research Center, Yorktown Heights, NY 10598.

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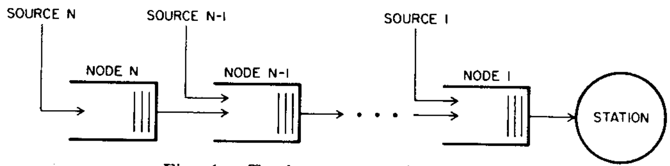

Fig. 1. Tandem system with N nodes.

Bernoulli arrival processes to the nodes. Simulations that were conducted show very good agreement with the proposed approximation.

II. N-Node Tandem: Packets Arrive Only at the Top Node

In this section, we consider an N-node tandem in which packets arrive only at node N. It is easy to see that the only node that can have more than one packet at any instant is node N. All other nodes $(1, 2, \cdots, N-1)$ can have at most one packet at a time.

Let $a(i)$ $i = 0, 1, 2, \cdots$ be the probability that i packets arrive at node N during a slot, and let $F(z) = \sum_{i=0}^{\infty} a(i)z^i$ be the p.g.f. of this arrival process. Let $r = \sum_{i=1}^{\infty} ia(i)$ be the arrival rate at node N. Then r is the arrival rate into each node of the tandem. Let the steady-state p.g.f. of the queue length at node N be $G_N(z)$. $G_N(z)$ can be obtained by analyzing node N as a single discrete-time queue with arrival process with p.g.f. $F(z)$ and a packet leaves the node (if any) every third slot. The result is:

$$G_N(z) = F(z) \frac{(1-3r)(1-z)}{F^3(z)-z} \tag{1}$$

and the condition for steady-state is that $r < 1/3$.

It is clear that the average number of packets at node $i(1 \leq i \leq N-1)$ is $\bar{L}_i = r$ and the average time delay at these nodes is one slot (no queues at the nodes). $\bar{L}_N$—the average number of packets at node N is obtained from (1):

$$\bar{L}_N = r + \frac{6r^2 + 3\sigma}{2(1-3r)} \tag{2}$$

where $\sigma = (d^2F(z)/dz^2)|_{z=1}$

Applying Little's law [4] to the whole tandem we obtain the average time delay in the system $-T$.

$$T = N + \frac{6r + 3\sigma/r}{2(1-3r)}. \tag{3}$$

The first term in (3) expresses the total time needed for a packet to traverse the tandem and the second term expresses the average waiting time at node N.

III. Four-Node Tandem: General Arrival Processes

In this section, we consider a four-node tandem when the arrival processes into the nodes are arbitrary. Let $A_i(t) 1 \leq i \leq 4$ $t = 0, 1, 2, \cdots$ be the number of packets entering node i from its corresponding source in the interval $(t, t+1)$. The joint input process $\{A_i(t)\}_{i=1}^{4}$ is assumed to be a sequence of independent and identically distributed random vectors with integer-valued elements. Notice that we allow the arrivals to different nodes to be dependent. Let the corresponding joint p.g.f. of the input arrival processes be $F(z) = E\{\prod_{i=1}^{4} z_i^{A_i(t)}\}$ where $z = (z_4, z_3, z_2, z_1)$.

A. Steady-State Analysis

Let $L_i(t)$ $1 \leq i \leq 4$ $t = 0, 1, 2, \cdots$ be the number of packets at node i at time t. Obviously, under our assumptions, $\{L_i(t)\}_{i=1}^{4}$ is a discrete-time discrete-state irreducible and aperiodic Markov chain. To describe its evolution we need the following. Let $U_i(t)$ $1 \leq i \leq 4$, $t = 0, 1, 2, \cdots$ be binary valued random variables that indicate whether node i has successfully transmitted a packet during slot t ($U_i(t) = 1$) or not ($U_i(t) = 0$). Consequently, if for notational convenience, we set $L_0(t) \equiv L_{-1}(t) \equiv 0$ for all t, we have for $1 \le i \le 4$, $t = 0, 1, 2, \cdots$:

$$U_i(t) = \begin{cases} 1 & L_{i-2}(t) = L_{i-1}(t) = 0,\ L_i(t) > 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

From the description of tandem systems it is easy to see now that for $t = 0, 1, 2, \cdots$ the system evolves according to the following equations:

$$L_i(t+1) = A_i(t) + L_i(t) - U_i(t) + U_{i+1}(t) \qquad 1 \le i \le 4 \tag{5}$$

where $U_5(t) \equiv 0$. In other words, (5) states that the number of packets at node i at the end of slot $t + 1$ is the sum of the number of packets at that node at the end of slot t and the number of packets arriving at the node during slot t. In addition, if node $i(i + 1)$ transmits a packet successfully during slot t, then the content of node i is decreased (increased) by 1.

Consider now the steady-state joint p.g.f. of the queue lengths $G(z) = \lim_{t\to\infty} E\{\Pi_{i=1}^{4} z_i^{L_i(t)}\}$. Here we assume that the Markov chain $\{L_i(t)\}_{i=1}^{4}$ is ergodic so that the above limit exists. In our case this assumption transforms to the condition that $G(z)|_{z_1=z_2=z_3=z_4=0} > 0$.
From (4) and (5), using a standard technique we obtain:

$$\begin{aligned} G(z_4, z_3, z_2, z_1) = F(z_4, z_3\, z_2, z_1)\{&G(0, 0, 0, 0) \\ &+[G(z_4, 0, 0, 0) - G(0, 0, 0, 0)]z_4^{-1}z_3 \\ &+[G(z_4, z_3, 0, 0,) - G(z_4, 0, 0, 0)]z_3^{-1}z_2 \\ &+[G(z_4, 0, 0, z_1) - G(z_4, 0, 0, 0) - G(0, 0, 0, z_1) \\ &+G(0, 0, 0, 0,)](z_4^{-1}z_3z_1^{-1} - z_1^{-1}) \\ &+[G(z_4, z_3, z_2, 0) - G(z_4, z_3, 0, 0)]z_2^{-1}z_1 \\ &+[G(z_4, z_3, z_2, z_1) - G(z_4, z_3, z_2, 0)]z_1^{-1}\}. \end{aligned} \tag{6}$$

The implication of each term in (6) should be clear; e.g., $G(z_4, 0, 0, 0) - G(0, 0, 0, 0)$ represents the case where nodes 1, 2, 3 are empty and node 4 is nonempty, and in such a case a packet is moved from node 4 to node 3 as shown in the term $z_4^{-1}z_3$. Other terms in the braces can be interpreted similarly. The factor $F(z_4, z_3, z_2, z_1)$ stands for changes in queue sizes due to the independent joint arrival process.

The complexity of the problem lies in the fact that in (6) $G(z)$ is expressed in terms of five boundary functions $G(z_4, 0, 0, 0)$, $G(z_4, z_3, 0, 0)$, $G(z_4, z_3, z_2, 0)$, $G(z_4, 0, 0, z_1)$ $G(0, 0, 0, z_1)$ and the constant $G(0, 0, 0, 0)$.The method for determining these unknown boundary terms is to use the analytic properties of the p.g.f. $G(z)$ in the polydisk $|z_i| < 1$ for $1 \le i \le 4$. Due to space limitations, we omit the cumbersome derivations and the interested reader is referred to [5]. We would just mention here that the condition for steady-state for a four-node tandem is $r_1 + 2r_2 + 3r_3 + 3r_4 < 1$ where $r_i = \partial F(z)/\partial z_i|_{z_1=z_2=z_3=z_4=1}$.

B. Average Queue Lengths and Time Delays

From the joint p.g.f. $G(z)$ any moment of the queue lengths at the various nodes can, in principle, be derived. Specifically, the average number of packets at node i ($1 \le i \le 4$) is $\bar{L}_i = \partial G(z)/\partial z_i|_{z_1=z_2=z_3=z_4=1}$. In addition, applying Little's law [4], we obtain the average time delay at node i($1 \le i \le 4$) which is $T_i = \bar{L}_i/\Sigma_{j=i}^{4} r_j$, since the total arrival rate at node i is $\Sigma_{j=i}^{4} r_j$. Finally, the total average delay in the system is $T = \Sigma_{i=1}^{4} \bar{L}_i/\Sigma_{i=1}^{4} r_i$.

IV. N-Node Tandem: Approximate Analysis

The analysis of a four-node network is quite complex because of the need to determine six boundary terms (five boundary functions and one boundary constant) in order to obtain the joint p.g.f. of the queue lengths at the nodes. Clearly the joint p.g.f. does not possess a product-form, therefore no decompositions are possible. It is also clear that the analysis will become much more complex (if possible at all), as the number of nodes increases, since the number of boundary terms to be determined will also increase.

To circumvent this difficulty, we propose here an approximate analysis method for obtaining average quantities such as queue lengths and time delays in an arbitrary tandem. The rationale behind the proposed approximation is that for approximating the behavior of a node in the tandem, it might suffice to take into account only those nearby nodes that directly affect that node. The effects of all other nodes of the tandem might be aggregated in some appropriate manner. To be more specific, the behavior of a node i in an N-node tandem system, can be approximated as the behavior of that node in a four-node tandem (and here is where we use our results from Section III) consisting of node i and its nearby nodes that directly affect it, namely node $i + 1$ that feeds it and nodes $i - 1$, $i - 2$ that interfere it. The effects of upstream nodes $i + 2, i + 3, \cdots, N$ and downstream nodes $1, 2, \cdots, i - 3$ will be aggregated by changing relevant parameters at nodes $i + 1$ and $i - 2$, respectively.

To formally introduce the approximate method, let us assume that the arrival processes into the nodes of the tandem are independent, namely $F(z) = \Pi_{i=1}^{N} F_i(z_i)$. In addition, in order to facilitate the presentation of the approximation, we further assume that the arrival process to node i is a Bernoulli process, i.e., $F_i(z_i) = z_i r_i + 1 - r_i$.

Let us now consider a node $i(i \ge 4)$ in a tandem of N nodes. We assume that $N > 4$ as for 4 nodes we obtained exact results. As explained above, the basic idea of the approximation is to use the results of Section III by considering nodes $i + 1, i, i - 1, i - 2$ as the four-node tandem and changing the arrival processes to nodes $i + 1$ and $i - 2$ so that the effects of all other nodes of the tandem on node i are captured. Specifically, we associate nodes $i + 1, i, i - 1, i - 2$ with nodes 4, 3, 2, 1 of Section III, respectively. The arrival process to node $i - 2$ should aggregate the effects on node i of packets traversing nodes $1, 2, \cdots, i - 3$. Accordingly, it is modified to,

$$F_{i-2}(z_{i-2}) = \prod_{j=1}^{i-1} F_j^{\delta_j(i)}(z_{i-2}) \tag{7}$$

where $\delta_1(i) = 1$; $\delta_2(i) = 2$ for all $i \ge 4$. $\delta_3(4) = 1$ and for all $i \ge 5$ $\delta_{i-1}(i) = 1$; $\delta_j(i) = 3$ $3 \le j \le i - 2$. The reason for (7) is that packets originated at nodes $i, i + 1, \cdots, N$, when traversing downstream nodes $i - 3, i - 4, \cdots, 1$, do not affect node i. $\delta_j(i)$ indicates the number of slots used solely by a packet originated at node j while traversing the downstream nodes. For instance, $\delta_6(4) = 3$ because after three slots, a packet originated at node 6 arrives to node 3 and then, while the packet is transmitted by node 3, another packet might be forwarded from node 6 to node 5.

To aggregate the effects of upstream nodes $i + 2, i + 3, \cdots, N$, the arrival process to node $i + 1$ is modified to:

$$F_{i+1}(z_{i+1}) = R_{i+1}z_{i+1} + 1 - R_{i+1} \tag{8}$$

where $R_{i+1} = \Sigma_{j=i+1}^{N} r_j$.

Remark: It is not too difficult to see that the proposed approximation coincides with the exact solution if packets arrive to the "top" node only.

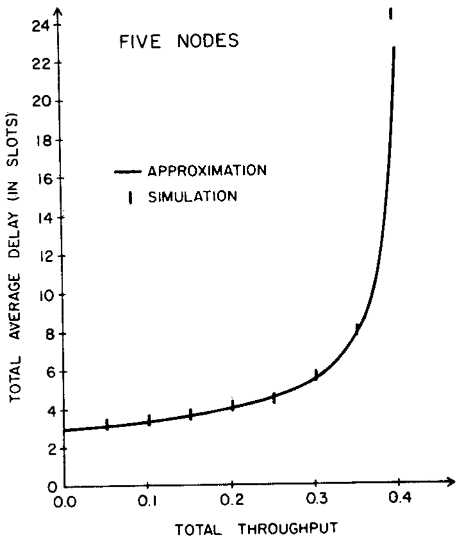

Fig. 2. Five-node tandem: Approximate analysis and simulation ($r_i = r$ $1 \leq i \leq 5$; total throughput $= 5r$).

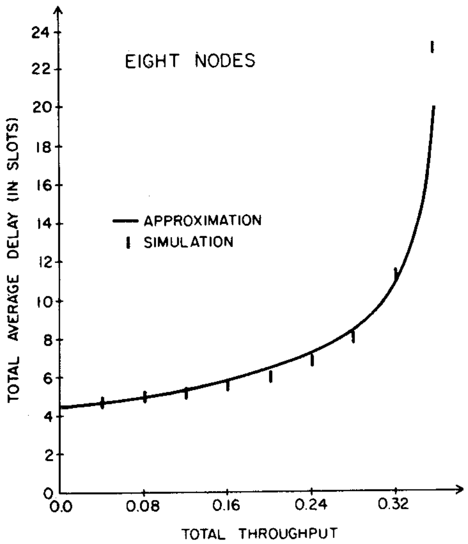

Fig. 3. Eight-node tandem: Approximate analysis and simulation ($r_i = r$ $1 \leq i \leq 8$; total throughput $= 8r$).

The results calculated by using our approximation are compared to those obtained from simulations in Fig. 2 for a 5-node tandem and in Fig. 3 for an 8-node tandem. Each point in the simulation represents operation of the tandem for 250 000 slots. In the tandems that we simulated we used $r_i = r$ for all i. In the figures we plotted the total average time delay as a function of γ the total throughput of the system. In Fig. 2 $\gamma = 5r$ and in Fig. 3 $\gamma = 8r$ where r is the arrival rate to each node of the tandem. The figures manifests very good agreement between our approximation and the conducted simulations for all throughput values.

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