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POLLING SYSTEMS WITH SYNCHRONIZATION CONSTRAINTS*

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Abstract

We introduce a new service discipline, called the *synchronized gated* discipline, for polling systems. It arises when there are precedence (or synchronization) constraints between the order that jobs in different queues should be served. These constraints are described as follows: There are N stations which are "fathers" of (zero or more) *synchronized stations* ("children"). Jobs that arrive at synchronized stations have to be processed only after jobs that arrived prior to them at their corresponding "father" station have been processed. We analyze the performance of the synchronized gated discipline and obtain expressions for the first two moments and the Laplace–Stieltjes transform (LST) of the waiting times in different stations, and expressions for the moments and LST of other quantities of interest, such as cycle duration and generalized station times. We also obtain a "pseudo" conservation law for the synchronized gated discipline, and determine the optimal network topology that minimizes the weighted sum of the mean waiting times, as defined in the "pseudo" conservation law. Numerical examples are given for illustrating the dependence of the performance of the synchronized gated discipline on different parameters of the network.

1. Introduction

The term *polling system* often refers in the literature to a system where a single server serves jobs from different queues (channels) in a predetermined order such as cyclic order or polling table with generally nonzero switchover times (see e.g. Takagi [15]). The service disciplines most often studied are the exhaustive, gated and limited service regimes:

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- Gated discipline: The server serves only those jobs which are waiting at the station when it is polled. The jobs that arrive during the service time are set aside to be served after the next polling instant to that station.
- Exhaustive discipline: The server serves all jobs in each queue until it is empty before moving to the next queue.
- Limited service: At each queue, service is completed if the number of jobs that have been served equals some threshold or if the queue empties before the threshold was attained.

We propose a new service discipline that arises when there are precedence (or synchronization) constraints between the order that jobs in different queues should be served. These constraints are described as follows: There are N stations which are "fathers" of (zero or more) *synchronized stations* ("children"). Jobs that arrive at synchronized stations have to be served only after jobs that arrived prior to them at their corresponding "father" station have been processed.

We suggest the following "synchronized gated" service discipline in order to meet these constraints: The server moves cyclically between the stations. When it arrives at a "father" station, it serves only those jobs which are waiting at the station when it is polled, as in the standard gated discipline. In addition, it marks upon arrival at that "father" station the jobs present in all its "children" stations. When it arrives at a synchronized station (a "child"), it serves only the jobs that were marked when the server last arrived at its "father" station. In both cases, jobs that are not served remain in the queues. Similarly to the standard gated or exhaustive schemes, the server keeps switching from one queue to the next even when there are no jobs present in the system.

An additional issue that is crucial for polling systems is the need for a service scheme that will allow designers to *prioritize* the different queues and thus affect and optimize overall system performance. In most polling models, it is common to prioritize the queues by controlling the amount of service given to each queue during the server's visit. Common service policies are the gated, exhaustive and limited service disciplines, each with its advantages and disadvantages. Other schemes are the binomial gated presented by Levy [11] and the globally gated scheme introduced recently by Boxma, Levy and Yechiali [7], which possess mechanisms for prioritizing the queues. Both schemes can be used to optimize the operation of the system at the design stage. In one sense, the "synchronized gated" scheme is another scheme for prioritizing the queues and optimize the operation of the system, and either the gated scheme or the globally gated scheme are both special cases of it. In the standard gated discipline, there are no synchronized stations. In the globally gated discipline, there is one selected station (one "father") such that at each polling instant to that station, gates are closed in all the stations (all other stations are synchronized with this "father"). In all the stations, the jobs that are served are only those that were present at the polling instant of the selected station. We note that the "synchronized gated" scheme is more flexible in prioritizing the queues to achieve a desired performance (see e.g. fig. 10 of numerical example number 3).

Different models for precedence constraints have been developed and studied in the past years, see e.g. [1–12]. Our paper is the first to analyze several precedence constraints that may appear in polling systems. These constraints exist when jobs that arrive at synchronized stations may need data obtained from jobs that arrived prior to them at their corresponding "father" station in order to be processed. For example, consider a polling system (say a token ring network) which consists of N nodes. Packets that arrive at each node belong to two different protocols (A) (say the TCP) and (B) (say the X.25). The requirements may be to serve packets which belong to protocol (A) before packets from protocol (B) in the whole network. This may be due to the large overhead required from the server when it switches from serving one protocol to the other protocol. That is, packets from protocol (A) have priority over packets from protocol (B). If we use the gated discipline to serve packets in the different queues, then this priority can be implemented in the following way: The service is given in two passes. In the first pass, when the server arrives at a node it closes a gate in that node and serves only packets from protocol (A) that were found in that node upon its arrival. In the second pass (the next polling instant), it serves only packets that belong to protocol (B) that it found when the gate was closed in the first pass. This scheme gives us the ability to prioritize sources in a *distributed* manner. Now, the "synchronized gated" scheme can be used to analyze such a two-passes system, where we define an equivalent network in which there are N "fathers" situated in subsequent positions in the network, and N "children" situated between "father" station N and "father" station 1. For each i, $i = 1, 2, \ldots, N$, the "father" of "child" station N_i is "father" station i. The arrival processes to the "father" and "child" stations are only the arrival processes from protocols (A) and (B), respectively, to the corresponding stations in the original system. The above arguments can be extended to any number of priority classes over the whole network. The above example shows the flexibility and the generality of the "synchronized gated" scheme over other schemes in describing prioritization in polling systems.

In section 8, we show that a priority polling system with vacations is a special case of the "synchronized gated" system. In this system, jobs in each node are served according to fixed priorities and the server takes vacations between the service of two subsequent priority classes.

Our main goal is to obtain expressions for the expected waiting times at the different stations. An efficient method to do so, that proved to be useful for the standard gated discipline (see e.g. [8,9]), is to first characterize the statistics of the "station times" (which are the times between the polling instants of two consecutive stations). This method relies on the fact that the vectors of the station times in the last cycle form a Markov chain. In our case, we define a "station time" to be the time between the polling instants of two consecutive "father" stations. The fact that we may restrict ourselves to these quantities (instead of defining station times for all stations) follows from the fact that "gates" are closed only when visiting a "father" station. This allows us to obtain a reduced number of equations for the Laplace transforms and for the second moments of the station times, and to obtain explicit expressions for the first moments. Unlike the standard gated discipline, the vectors of the station times in the last cycle do not form a Markov chain. We overcome this problem by enlarging the state vector to include station times of the last two cycles.

We obtain a "pseudo" conservation law for the "synchronized gated" discipline, and determine the optimal network topology that minimizes the weighted sum of the mean waiting times obtained in the "pseudo" conservation law.

Among other things, we show that for any station the expectation of the number of jobs present in its queue when the station starts to be served is greater than (for "child" station) or equal to (for "father" station) the expectation obtained for the standard gated service discipline.

We faced a nontrivial problem in the notation when trying to enumerate the different stations in a way that grasps both the (cyclic) order in which they are served as well as the order according to the precedence constraints. We present an enumeration that partially meets these goals and yet is simple and convenient for the subsequent analysis.

The structure of the paper is as follows. In section 2, we describe the model together with the assumptions and several definitions and notations used throughout the paper. In section 3, we describe the evolution equations of the system and derive a recursion equality for the joint generating functions of the station times. In section 4, we derive the first moments of the station times in steady state, and obtain a set of $N(2N-1)$ linear equations from which the second moments of the station times in steady state can be computed. These moments are then used in section 5 to obtain the average waiting times in the different stations of the network. In section 5, we derive the Laplace transform and the expectation of the waiting times for jobs in "father" and in "child" stations. In section 6, we derive a "pseudo" conservation law for a polling system with synchronization constraints and compare it with the standard gated discipline. We also determine the optimal network topology with respect to the conservation law. In section 7, we give three numerical examples of polling systems with synchronization constraints for illustrating the dependence of the performance of the "synchronized gated" discipline on the different parameters of the network. In section 8, we summarize and discuss possible extensions and applications of our results.

2. The model and notations

The polling system has M stations, of which N are "fathers" and the rest are "children". We label some arbitrary father station by 1. We then enumerate the rest of the father stations according to the (cyclic) order of service between them; they are thus given the numbers $2, \ldots, N$. Let $m(i)$ denote the number of child stations that are situated between (the father) station i and the subsequent father station. These stations are indexed by the set $\{i_j, j = 1, \ldots, m(i)\}$; the subscript j is given according to the order in which they are served. Sometimes, for convenience we shall use i_0 to denote the father station i. For each station i_j, $j \neq 0$, we denote by $s(i_j)$ the father of that station, and we set $s(i_0) = i$.

The stations are served as follows. The server moves cyclically between the stations. When it arrives at a father station, it serves only those jobs which are waiting at the station when it is polled, as in the standard gated discipline. In addition, it marks upon arrival to that father station the jobs present in all its children stations. When it arrives at a synchronized station (a child), it serves only the jobs that were marked when the server last arrived at its father station. In both cases, jobs that are not served remain in the queues. Similarly to the standard gated or exhaustive schemes, the server keeps switching from one queue to the next even when there are no jobs present in the system.

The queue capacity at each station is infinite. The arrival process at each station i_j is independent and Poisson distributed with arrival rate λ_{i_j}. The service times are all independent; their generating function in the i_jth station is denoted by $B^*_{i_j}(s)$, and first and second moments are denoted by $\bar{b}_{i_j}$ and $b^{(2)}_{i_j}$, respectively. The time it takes between the end of service in station i_j until the polling instant in the next station is called the i_jth *walking time*. Let d_{i_j} be a random variable (RV) that stands for this time. We assume that the walking times are independent and their Laplace transform, first and second moments are denoted by $D^*_{i_j}(s)$, $\bar{d}_{i_j}$ and $d^{(2)}_{i_j}$, respectively; moreover, these times, the interarrival times and the service durations are mutually independent. We further assume that for each station i_j, $1 \leq i \leq N$, $0 \leq j \leq m(i)$, its first and second moments of the service times and the walking times are finite.

Let

$$\rho_{i_j} \overset{\text{def}}{=} \lambda_{i_j} \bar{b}_{i_j} \quad \text{and} \quad \rho \overset{\text{def}}{=} \sum_{i=1}^{N} \sum_{j=0}^{m(i)} \rho_{i_j}.$$

Throughout this paper, we make the standard stability assumption, i.e. $\rho < 1$.

In order to obtain the evolution equations for the system, we look at the random times $\tau(k)$, which represent the kth polling instant of a father station. Thus, a father station that is polled at $\tau(k)$ is also polled at times $\tau(k+N)$, $\tau(k+2N)$, and so on. If station i is polled at time $\tau(k)$, we define by $\tau(k_j)$ the first time that station i_j is polled after time $\tau(k)$. With some abuse of notation, we shall refer to station k (and k_j) as the station that is visited at time $\tau(k)$ ($\tau(k_j)$, respectively). If $s(k_j) = i$, then $\hat{s}(k_j)$ will denote the largest time index $k' \leq k$ such that station i is visited at time $\tau(k')$. The quantity $\tau(\hat{s}(k_j))$ is thus the largest polling time of station $s(k_j)$ before $\tau(k_j)$. With some abuse of notation, we shall use $m(k)$ to denote the number of child stations that are situated between (the father) station visited at $\tau(k)$ and the subsequent father station.

Generalizing the definitions for the standard gated discipline [8], we define for any father station k the *station time* θ_k as the (random) time it takes to serve that station as well as all the subsequent child stations $k_j, j = 1, \ldots, m(k)$, plus the overall walking time between the kth and the $(k+1)$st station.

Let $d(k)$ be the RV that represents the total walking time between the father station k to the next father station. Its generating function is denoted by $D_k^*(s)$. For a given $i = 1, 2, \ldots, N$, note that $d(i + lN)$, $l = 0, 1, 2, \ldots$, are i.i.d. RVs. Let d_{i_f} (f stands for a father station) denote a generic RV with that distribution. Its generating function is denoted by $D_{i_f}^*(s)$, and the first and second moments are denoted by $\bar{d}_{i_f}$ and $d_{i_f}^{(2)}$, respectively. Note that $D_{i_f}^*(s) = \prod_{j=0}^{m(i)} D_{i_j}^*(s)$. Let

$$d \overset{\text{def}}{=} \sum_{i=1}^{N} \sum_{j=0}^{m(i)} d_{i_j}, \quad \bar{d} \overset{\text{def}}{=} E[d], \quad d^{(2)} \overset{\text{def}}{=} E[d^2].$$

3. The generating functions

3.1. THE EVOLUTION EQUATIONS

The *state* of the system at the kth polling instant is the following vector of $2N-1$ station times:

$$\vartheta_k = \{\theta_{k-(2N-2)}, \theta_{k-(2N-3)}, \ldots, \theta_{k-1}, \theta_k\}. \tag{1}$$

Comparing this choice of state to the one in the standard gated discipline (see [8]), we note that whereas in the standard gated discipline only station times from the previous "cycle" are considered, we need here station times from the two previous cycles (the number of station times considered may be reduced, as explained later in section 8). The need for those will become clear in the anaysis to follow. However, we do not need to define and incorporate station times for all the M stations in order to obtain the dynamics of the system in father stations.

To describe the evolution of the system at the polling instants, we need the following definitions. Let T_l^k be an RV that stands for the time it takes to serve station k_l and denote $T(k) \overset{\text{def}}{=} \sum_{l=0}^{m(k)} T_l^k$. Note that T_j^k is the time it takes to serve the jobs that arrived at station k_j during the station times $\theta_{\hat{s}(k_j)-N}, \theta_{\hat{s}(k_j)-N+1}, \ldots, \theta_{\hat{s}(k_j)-1}$. This fact motivated the incorporation of station times from more than one cycle into the state of the system. To obtain the state of the system at the $(k+1)$st polling instant, we need to compute the $(k+1)$st station time which is given by the overall service times of the stations which were visited by the kth polling instant and the $(k+1)$st polling instant (given by $T(k+1)$) and the overall walking times between these polling instants (given by $d(k+1)$). Then, the state of the system at the $(k+1)$st polling instant is:

$$\vartheta_{k+1} = \{\theta_{k-(2N-3)}, \theta_{k-(2N-4)}, \ldots, \theta_{k-1}, \theta_k, T(k+1) + d(k+1)\}.$$

Note that the values of $2N-2$ components of ϑ_k and of ϑ_{k+1} are common.

3.2. THE JOINT GENERATING FUNCTIONS OF THE STATION TIMES

For $\boldsymbol{x} \stackrel{\text{def}}{=} \{x_0, x_1, \ldots, x_{2N-2}\}$, define

$$G_k^*(\boldsymbol{x}) \stackrel{\text{def}}{=} E \exp\left[-\sum_{j=0}^{2N-2} x_j\,\theta_{k-j}\right],$$

where x_i, $0 \le i \le 2N-2$, are variables having nonnegative real parts. Also define the following indicator functions:

$$I^k(j,n) \stackrel{\text{def}}{=} \begin{cases} 1, & k-\hat{s}(k_j) \le n \le k-\hat{s}(k_j)+N-1, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

$$n = 0, 1, \ldots, 2N-2, \quad j = 0, 1, \ldots, m(k).$$

LEMMA 1

The generating functions $G_k^*(\boldsymbol{x})$, $k = 1, 2, \ldots$ satisfy the following recursion:

$$G_{k+1}^*(\boldsymbol{x}) = D_{k+1}^*(x_0) G_k^*(\boldsymbol{\alpha}), \tag{3}$$

where $\boldsymbol{\alpha} \stackrel{\text{def}}{=} \{\alpha_0, \alpha_1, \ldots, \alpha_{2N-2}\}$ is given by:

$$\alpha_n = x_{n+1} \cdot \mathbf{1}\{n \ne 2N-2\} + \sum_{j=0}^{m(k+1)} I^{k+1}(j,n)\lambda_{(k+1)_j}\left(1 - B^*_{(k+1)_j}(x_0)\right).$$

Proof

$$\begin{aligned} G_{k+1}^*(\boldsymbol{x}) &= E \exp\left[-\sum_{j=0}^{2N-2} x_j\,\theta_{k+1-j}\right] \\ &= E \exp\left[-\sum_{j=1}^{2N-2} x_j\,\theta_{k-j+1} - x_0(T(k+1) + d(k+1))\right] \\ &= E \exp[-x_0 d(k+1)] E\left\{E\left(\exp\left[-\sum_{j=1}^{2N-2} x_j\,\theta_{k-j+1} - x_0 T(k+1)\right] \Bigg| \vartheta_k\right)\right\} \\ &= D_{k+1}^*(x_0) E\left\{\exp\left[-\sum_{j=1}^{2N-2} x_j\,\theta_{k-j+1}\right] E\left(\exp\left[-x_0 T(k+1)\right] \big| \vartheta_k\right)\right\}. \end{aligned}$$

Since $T(k+1) \stackrel{\text{def}}{=} \sum_{j=0}^{m(k+1)} T_j^{k+1}$ and since the summands are independent given ϑ_k, we obtain:

$$G_{k+1}^*(\boldsymbol{x}) = D_{k+1}^*(x_0) E\left\{ \exp\left[-\sum_{j=1}^{2N-2} x_j \,\theta_{k-j+1} \right] \prod_{j=0}^{m(k+1)} E\left(\exp\left[-x_0 T_j^{k+1} \right] \middle| \vartheta_k \right) \right\}. \quad (4)$$

The last expression can be evaluated by direct integration (see cf. [8] for the more simple case of standard exhaustive discipline). However, this cumbersome calculation can be avoided by recalling the well-known solution of the following auxiliary problem.

Auxiliary problem: Assume a Poisson flow of jobs with rate λ. Let T be the (random) time that it takes to serve those jobs which arrived during the random time L whose Laplace transform is $L^*(s)$. The service times of different jobs are i.i.d. random variables with Laplace transform $B^*(s)$; these times, the interarrival times and L are mutually independent. Then the Laplace transform of the distribution of T is given by

$$T^*(s) = E\exp\{-sT\} = L^*(\lambda - \lambda B^*(s))$$

and as a special case, $E(\exp\{-sT\} \mid L = l) = \exp\{-(\lambda - \lambda B^*(s))l\}$. Using this fact, we obtain from (4):

$$G_{k+1}^*(\boldsymbol{x}) = D_{k+1}^*(x_0) E\left\{ \exp\left[-\sum_{j=1}^{2N-2} x_j \,\theta_{k-j+1} \right] \right.$$

$$\left. \cdot \prod_{j=0}^{m(k+1)} \exp\left[-\lambda_{(k+1)_j} \left(1 - B_{(k+1)_j}^*(x_0)\right) \sum_{l=1}^{N} \theta_{\hat{s}((k+1)_j)-l} \right] \right\}$$

$$= D_{k+1}^*(x_0) E \exp\left\{ -\sum_{j=1}^{2N-2} x_j \,\theta_{k-j+1} \right.$$

$$\left. -\sum_{j=0}^{m(k+1)} \left[\lambda_{(k+1)_j} \left(1 - B_{(k+1)_j}^*(x_0)\right) \sum_{l=1}^{N} \theta_{\hat{s}((k+1)_j)-l} \right] \right\}.$$

The theorem then follows by using the definition in (2), and the α_n are the coefficients that multiply the appropriate θ's in the last expression. □

Next, we obtain the steady-state statistical behavior of the vectors of station times. In what follows, quantities in steady state are denoted by a tilde. Assume that at $\tau(1)$ station 1 was polled. Under standard stability conditions ($\rho < 1$), for each i, $1 \le i \le N$, ϑ_{i+lN} converge in distribution to some random vector $\tilde{\vartheta}_i$ as $l \to \infty$; we denote its components by $\tilde{\theta}_{i-j}, j = 0, \ldots, 2N-2$ and the corresponding Laplace transform by $\tilde{G}_i(\boldsymbol{x})$, i.e.

$$\tilde{G}_i(\boldsymbol{x}) \stackrel{\text{def}}{=} E \exp\left[-\sum_{j=0}^{2N-2} x_j \tilde{\theta}_{i-j} \right].$$

Note that for each i, $1 \le i \le N$, $\tilde{\theta}_{i-N}$ has the same distribution as $\tilde{\theta}_i$.

In order to obtain recursive expressions for $\tilde{G}_i(\boldsymbol{x})$, we define the following indicator functions. Let $i = 1, 2, \ldots, N$, $n = 0, 1, \ldots, 2N-2$, $j = 0, 1, \ldots, m(i)$. If $s(i_j) \le i$, then

$$\tilde{I}^i(j,n) = \begin{cases} 1, & i - s(i_j) \le n \le i - s(i_j) + N - 1, \\ 0, & \text{otherwise.} \end{cases}$$

If $s(i_j) > i$, then

$$\tilde{I}^i(j,n) = \begin{cases} 1, & i + N - s(i_j) \le n \le i - s(i_j) + 2N - 1, \\ 0, & \text{otherwise.} \end{cases}$$

THEOREM 2

The generating functions $\tilde{G}_i(\boldsymbol{x})$, $i = 1, \ldots, N$, can be computed by solving the following set of equations:

$$\tilde{G}_{i+1}(\boldsymbol{x}) = D^*_{(i+1)_f}(x_0)\tilde{G}_i(\tilde{\boldsymbol{\alpha}}), \qquad i = 1, 2, \ldots, N, \tag{5}$$

where $\tilde{\boldsymbol{\alpha}} \stackrel{\text{def}}{=} \{\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_{2N-2}\}$ is given by:

$$\tilde{\alpha}_n = x_{n+1} \cdot \mathbf{1}\{n \ne 2N-2\} + \sum_{j=0}^{m(i+1)} \tilde{I}^{i+1}(j,n)\lambda_{(i+1)_j}\left(1 - B^*_{(i+1)_j}(x_0)\right). \tag{6}$$

Proof

Follows immediately from lemma 1. □

Remark

In (5) and (6), we understand by $i+1$ to be equal to 1 when $i = N$.

4. The moments

4.1. THE FIRST MOMENTS

Let C_{i_j} be the RV which represents the i_jth cycle duration in steady-state regime, i.e. the time it takes between the polling of station i_j until the next time it is polled. Clearly, $E[C] \stackrel{\text{def}}{=} E[C_{i_j}]$ does not depend on i or j. Let $c(k)$ be the RV which represents the kth cycle duration. Let $\delta(k) \stackrel{\text{def}}{=} \sum_{n=0}^{N-1} d(kN+n)$, i.e. the total walking time during the kth cycle.

PROPOSITION 3

The expectation $E[C]$ satisfies

$$E[C] = \frac{\bar{d}}{1-\rho} \tag{7}$$

for ergodic cyclic polling systems with arbitrary service discipline, provided that

(i) the server does not idle (except for the walking times),

(ii) with probability one,

$$\lim_{k\to\infty} \frac{1}{k} \sum_{l=1}^{k} c(l) = E[C], \quad \lim_{k\to\infty} \frac{1}{k} \sum_{l=1}^{k} \delta(l) = \bar{d}.$$

Proof

Since the system is ergodic ($\rho < 1$), and an amount of work ρ per time unit is offered to the server, we have that the fraction of time during which the server is busy is equal to ρ. We show below that the fraction of time during which the server is not serving is equal to $\bar{d}/E[C]$, from which we obtain

$$1-\rho = \frac{\bar{d}}{E[C]},$$

which establishes the proposition. It remains to show that the fraction of time during which the server is not serving is equal to $\bar{d}/E[C]$. The quantity is given by

$$\frac{\lim_{k\to\infty} \sum_{l=1}^{k} \delta(l)}{\lim_{k\to\infty} \sum_{l=1}^{k} c(l)} = \frac{\lim_{k\to\infty} \frac{1}{k} \sum_{l=1}^{k} \delta(l)}{\lim_{k\to\infty} \frac{1}{k} \sum_{l=1}^{k} c(l)} = \frac{\bar{d}}{E[C]}.$$

□

Conditions (i) and (ii) of the proposition clearly hold in our case.

We are now ready to calculate the expectations of the station times:

$$E[\theta_k] = E\{E[\theta_k \mid \vartheta_{k-1}]\} = E[d(k)] + \sum_{j=0}^{m(k)} \rho_{k_j} \sum_{l=1}^{N} E\left[\theta_{\hat{s}(k_j)-l}\right],$$

which in the steady-state regime yields

$$E[\tilde{\theta}_i] = \bar{d}_{i_f} + \sum_{j=0}^{m(i)} \rho_{i_j} E[C] = \bar{d}_{i_f} + \bar{\rho}_i \frac{\bar{d}}{1-\rho}, \qquad i = 1, 2, \ldots, N, \tag{8}$$

where in (8) we define

$$\bar{\rho}_i \stackrel{\text{def}}{=} \sum_{j=0}^{m(i)} \rho_{i_j}.$$

Note that for each i, $i = 1, 2, \ldots, N$, $E[\tilde{\theta}_{i-l}]$ is defined for $l = 1, 2, \ldots, 2N-2$, where

$$E[\tilde{\theta}_{i-l}] = \begin{cases} E[\tilde{\theta}_{N+i-l}]; & -(N-1) \le i-l \le 0, \\ E[\tilde{\theta}_{2N+i-l}]; & -(2N-3) \le i-l \le -N. \end{cases} \tag{9}$$

4.2. THE SECOND MOMENTS

In what follows, we shall use the following notation: For two integers i and j, let

$$i \ominus j \stackrel{\text{def}}{=} \begin{cases} i-j, & i \ge j, \\ i-j+2N, & \text{otherwise.} \end{cases} \tag{10}$$

Define for $i = 1, \ldots, N$, $j = 1, \ldots, 2N$, the following second central moments,

$$r_{ij} \stackrel{\text{def}}{=} \begin{cases} E[(\tilde{\theta}_i - E\tilde{\theta}_i)(\tilde{\theta}_j - E\tilde{\theta}_j)]; & j \le i, \\ E[(\tilde{\theta}_i - E\tilde{\theta}_i)(\tilde{\theta}_{j-2N} - E\tilde{\theta}_j)]; & j > i+1, \end{cases} \tag{11}$$

and define

$$\rho_i(j) \stackrel{\text{def}}{=} \sum_{l=0}^{m(i)} \tilde{I}^i(l, i \ominus (j+1)) \rho_{i_l}. \tag{12}$$

We further define for $i = N+1, N+2, \ldots, 2N$, $i - j \ne 2N-1$

$$r_{ij} \stackrel{\text{def}}{=} \begin{cases} r_{(i-N)(j+N)}; & 1 \le j \le N, \\ r_{(i-N)(j-N)}; & N+1 \le j \le 2N. \end{cases} \tag{13}$$

Note that, in the above definition of r_{ij}, $\tilde{\theta}_j$ corresponds to a station time that *comes before* the station time $\tilde{\theta}_i$. Also, note that r_{ij} and r_{ji} need not be equal.

THEOREM 4

Using the definitions in (10)–(13), the second moments of the station times in steady-state regime can be calculated by solving the following $N(2N-1)$ equations:

$$r_{ij} = \begin{cases} \sum_{m=i+1}^{2N} \rho_i(m) r_{jm} + \sum_{m=1}^{j-1} \rho_i(m) r_{jm} + \sum_{m=j}^{i-1} \rho_i(m) r_{mj}; & j < i, \\ \sum_{m=i+1}^{j-1} \rho_i(m) r_{jm} + \sum_{m=j}^{2N} \rho_i(m) r_{mj} + \sum_{m=1}^{i-1} \rho_i(m) r_{mj}; & j > i+1; \end{cases} \tag{14}$$

$$\begin{aligned} r_{ii} = \operatorname{var}(d_{i_f}) &+ \sum_{n=0}^{2N-2} \left(\sum_{l=0}^{m(i)} \tilde{I}^i(l,n) \lambda_{i_l} b_{i_l}^{(2)} \right) E[\tilde{\theta}_{i-n-1}] \\ &+ \sum_{\substack{j=1 \\ j \neq i, i+1}}^{2N} \rho_i(j) r_{ij} + \rho_i(i+1) \sum_{\substack{j=1 \\ j \neq i}}^{2N} \rho_i(j) r_{j(i+1)} \end{aligned} \tag{15}$$

$$i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, 2N.$$

The proof of theorem 4 appears in appendix A.

5. The waiting times

In this section, we assume that service in each queue is given according to the first-come-first-serve (FCFS) discipline. Standard techniques can be used to compute the Laplace transform and the expectation of the waiting times for jobs in father stations, using the previous expressions for the Laplace transform of the station times (see e.g. [8] or [9]). For the synchronized stations the situation is more complex, and we obtain the distribution of the waiting times through a similar method to the one used in [15].

5.1. WAITING TIMES IN THE FATHER STATIONS

Let W_i denote the waiting time at an arbitrary moment in steady-state regime in the father station i, $i = 1, 2, \ldots, N$, and let $W_i^*(s)$ denote its Laplace transform. Let $C_i^*(s)$ denote the Laplace transform of the ith cycle duration in steady-state C_i, $i = 1, \ldots, N$. We have:

$$C_i^*(s) \stackrel{\text{def}}{=} E[\exp\{-sC_i\}] = E\left[\exp\left(-s\sum_{l=1}^{N}\tilde{\theta}_{i-l}\right)\right]$$

$$= \tilde{G}_i(x)\Big|_{\substack{x_0 = x_{N+1} = x_{N+2} = \ldots = x_{2N-2} = 0 \\ x_1 = x_2 = \ldots = x_N = s}} . \tag{16}$$

The Laplace transform of the waiting time is then given by

$$W_i^*(s) = \frac{1}{E[C]}\,\frac{C_i^*[\lambda_i - \lambda_i B_i^*(s)] - C_i^*(s)}{s - \lambda_i + \lambda_i B_i^*(s)}$$

and the expectation of the waiting time is given by

$$E[W_i] = E[C]\,\frac{1+\rho_i}{2} + \frac{(1+\rho_i)\,\mathrm{var}(C_i)}{2E[C]},$$

where

$\mathrm{var}(C_i)$

$$= E\left[\left(\sum_{l=1}^{N}\left(\tilde{\theta}_{i-l} - E\tilde{\theta}_{i-l}\right)\right)^2\right] = \sum_{n=i-N}^{i-1}\ \sum_{m=i-N}^{i-1} E\left[\left(\tilde{\theta}_n - E\tilde{\theta}_n\right)\left(\tilde{\theta}_m - E\tilde{\theta}_m\right)\right]$$

$$= \sum_{n=1}^{i-1}\left(\sum_{m=i+N}^{2N} r_{nm} + \sum_{m=1}^{n-1} r_{nm} + \sum_{m=n}^{i-1} r_{mn}\right) + \sum_{n=i+N}^{2N}\left(\sum_{m=i+N}^{n-1} r_{nm} + \sum_{m=n}^{2N} r_{mn} + \sum_{m=1}^{i-1} r_{mn}\right).$$

5.2. WAITING TIMES IN THE SYNCHRONIZED STATIONS

In this subsection, we compute the Laplace transform and the expectation of the waiting times for jobs in the synchronized stations, namely, for the stations:

$$i_j, \quad i = 1, 2, \ldots, N, \quad j = 1, \ldots, m(i).$$

The formulas we obtain for these quantities hold also for jobs in the father stations. In order to proceed, we define the following random variables in steady-state regime:

L_{i_j} $\stackrel{\text{def}}{=}$ the number of jobs that a job departing from station i_j sees at station i_j;
W_{i_j} $\stackrel{\text{def}}{=}$ the waiting time at an arbitrary moment in station i_j;
$X_{i_j}^{i_j}$ $\stackrel{\text{def}}{=}$ the number of jobs in station i_j at the polling instant to that station;
$X_{s(i_j)}^{i_j}$ $\stackrel{\text{def}}{=}$ the number of jobs in station i_j at the polling instant to its father $s(i_j)$.

Let $\tilde{s}(i_j) \stackrel{\text{def}}{=} s(i_j) - N \cdot \mathbf{1}\{s(i_j) > i\}$, and define an empty sum to equal zero. We define the RV $d_{i \to j}$ to be the total walking time from station i to station i_j. Let $D_{i \to j}^*(s)$ denote its LST. Then, $D_{i \to j}^*(s) = \prod_{n=0}^{j-1} D_{i_j}^*(s)$. Define the moment generating function

$$Q_{i_j}(z) \stackrel{\text{def}}{=} E\left[z^{L_{i_j}}\right].$$

In appendix B, we compute $Q_{i_j}(z)$ and obtain

$$Q_{i_j}(z) = \frac{B_{i_j}^*(\lambda_{i_j} - \lambda_{i_j} z)}{E\left[X_{s(i_j)}^{i_j}\right]} \frac{E\left[\left(B_{i_j}^*(\lambda_{i_j} - \lambda_{i_j} z)\right)^{X_{s(i_j)}^{i_j}} \cdot z^{X_{i_j}^{i_j} - X_{s(i_j)}^{i_j}}\right] - E\left[z^{X_{i_j}^{i_j}}\right]}{B_{i_j}^*(\lambda_{i_j} - \lambda_{i_j} z) - z}. \tag{17}$$

The waiting time distribution is related to $Q_{i_j}(z)$ through

$$Q_{i_j}(z) = W_{i_j}^*(\lambda_{i_j} - \lambda_{i_j} z) B_{i_j}^*(\lambda_{i_j} - \lambda_{i_j} z). \tag{18}$$

Note also that

$$E\left[X_{s(i_j)}^{i_j}\right] = E\left[E\left(X_{s(i_j)}^{i_j} \mid \tilde{\vartheta}_{s(i_j)-1}\right)\right]$$
$$= E\left[\lambda_{i_j} \sum_{l=\tilde{s}(i_j)-1}^{\tilde{s}(i_j)-N} \tilde{\theta}_l\right] = \lambda_{i_j} E\left[C_{s(i_j)}\right] = \lambda_{i_j} E[C]. \tag{19}$$

Using (17)–(19), we obtain

$$W_{i_j}^*(s) = \frac{1}{E[C]} \frac{E\left[\left(B_{i_j}^*(\lambda_{i_j} - \lambda_{i_j} z)\right)^{X_{s(i_j)}^{i_j}} \cdot z^{X_{i_j}^{i_j} - X_{s(i_j)}^{i_j}}\right]_{z=1-s/\lambda_{i_j}} - E\left[z^{X_{i_j}^{i_j}}\right]_{z=1-s/\lambda_{i_j}}}{s - \lambda_{i_j} + \lambda_{i_j} B_{i_j}^*(s)}. \tag{20}$$

In what follows, we calculate the terms which appear in the numerator of (20).

- $$E\left[\left(B_{i_j}^*(\lambda_{i_j}-\lambda_{i_j}z)\right)^{X_{s(i_j)}^{i_j}}\cdot z^{X_{i_j}^{i_j}-X_{s(i_j)}^{i_j}}\right]$$

$$=E\left(E\left[\left(B_{i_j}^*(\lambda_{i_j}-\lambda_{i_j}z)\right)^{X_{s(i_j)}^{i_j}}\cdot z^{X_{i_j}^{i_j}-X_{s(i_j)}^{i_j}}\middle|\tilde{\vartheta}_{i-1}\right]\right)$$

$$=D_{i\to j}^*(\lambda_{i_j}-\lambda_{i_j}z)E\left(\exp\left\{-\lambda_{i_j}(1-z)\sum_{n=\tilde{s}(i_j)}^{i-1}\tilde{\theta}_n\right.\right.$$

$$\left.-\sum_{l=0}^{j-1}\left[\lambda_{i_l}\left(1-B_{i_l}^*(\lambda_{i_j}-\lambda_{i_j}z)\right)\sum_{n=1}^{N}\tilde{\theta}_{\tilde{s}(i_l)-n}\right]\right\}$$

$$\left.\times\exp\left\{-\lambda_{i_j}\left(1-B_{i_j}^*(\lambda_{i_j}-\lambda_{i_j}z)\right)\sum_{n=1}^{N}\tilde{\theta}_{\tilde{s}(i_j)-n}\right\}\right).$$

The last equality is obtained using the auxiliary problem at the end of this section. We thus obtain:

$$E\left[\left(B_{i_j}^*(\lambda_{i_j}-\lambda_{i_j}z)\right)^{X_{s(i_j)}^{i_j}}\cdot z^{X_{i_j}^{i_j}-X_{s(i_j)}^{i_j}}\right]=D_{i\to j}^*(\lambda_{i_j}-\lambda_{i_j}z)\tilde{G}_{i-1}^*\ (\boldsymbol{\beta}),$$

where $\boldsymbol{\beta}=\{\beta_0,\beta_1,\ldots,\beta_{2N-2}\}$ is given by

$$\beta_n=\mathbf{1}\{0\le n\le i-\tilde{s}(i_j)-1\}\cdot\lambda_{i_j}(1-z)+\sum_{l=0}^{j-1}\tilde{I}^i(l,n)\lambda_{i_l}\left(1-B_{i_l}^*(\lambda_{i_j}-\lambda_{i_j}z)\right)$$

$$+\mathbf{1}\{i-\tilde{s}(i_j)\le n\le i-\tilde{s}(i_j)+N-1\}\cdot\lambda_{i_j}\left(1-B_{i_j}^*(\lambda_{i_j}-\lambda_{i_j}z)\right)\qquad(21)$$

$$n=0,1,\ldots,2N-2.$$

- $$E\left[z^{X_{i_j}^{i_j}}\right]=D_{i\to j}^*(\lambda_{i_j}-\lambda_{i_j}z)E\exp\left\{-\lambda_{i_j}(1-z)\sum_{n=\tilde{s}(i_j)-N}^{i-1}\tilde{\theta}_n\right.$$

$$\left.-\sum_{l=0}^{j-1}\left[\lambda_{i_l}\left(1-B_{i_l}^*(\lambda_{i_j}-\lambda_{i_j}z)\right)\sum_{n=1}^{N}\tilde{\theta}_{\tilde{s}(i_l)-n}\right]\right\},$$

from which we obtain:

$$E\left[z^{X^{i_j}_{i_j}}\right] = D^*_{i\to j}(\lambda_{i_j} - \lambda_{i_j} z)\tilde{G}^*_{i-1}(\boldsymbol{\gamma}),$$

where $\boldsymbol{\gamma} = \{\gamma_0, \gamma_1, \ldots, \gamma_{2N-2}\}$ is given by

$$\begin{aligned}\gamma_n &= \mathbf{1}\{0 \le n \le i - \tilde{s}(i_j) + N - 1\} \cdot \lambda_{i_j}(1-z) \\ &\quad + \sum_{l=0}^{j-1} \tilde{I}^i(l,n)\lambda_{i_l}\left(1 - B^*_{i_l}(\lambda_{i_j} - \lambda_{i_j} z)\right)\end{aligned}$$

$$n = 0, 1, \ldots, 2N-2.$$

Next, we calculate the *expected* waiting time in a synchronized station:

$$\begin{aligned}E[W_{i_j}] &= -\frac{\mathrm{d}}{\mathrm{d}s} W^*_{i_j}(s)\bigg|_{s=0} \\ &= \frac{(1+\rho_{i_j})\mathrm{var}\left(X^{i_j}_{s(i_j)}\right)}{2\lambda_{i_j} E\left[X^{i_j}_{s(i_j)}\right]} + \frac{E\left[X^{i_j}_{s(i_j)}\left(X^{i_j}_{i_j} - X^{i_j}_{s(i_j)}\right)\right] - E\left[X^{i_j}_{s(i_j)}\right]E\left[X^{i_j}_{i_j} - X^{i_j}_{s(i_j)}\right]}{\lambda_{i_j} E\left[X^{i_j}_{s(i_j)}\right]} \\ &\quad + \frac{E\left[X^{i_j}_{i_j} - X^{i_j}_{s(i_j)}\right]}{\lambda_{i_j}} + \frac{(1+\rho_{i_j})E\left[X^{i_j}_{s(i_j)}\right]}{2\lambda_{i_j}} - \frac{(1+\rho_{i_j})}{2\lambda_{i_j}},\end{aligned}$$

where $E[X^{i_j}_{s(i_j)}]$ is given in (19), and

- $$\begin{aligned}E\left[X^{i_j}_{i_j} - X^{i_j}_{s(i_j)}\right] &= E\left[E\left(X^{i_j}_{i_j} - X^{i_j}_{s(i_j)} \middle| \tilde{\vartheta}_{i-1}\right)\right] \\ &= E\left[\lambda_{i_j} \sum_{l=\tilde{s}(i_j)}^{i-1} \tilde{\theta}_l + \lambda_{i_j}\left(E[d_{i\to j}] + \sum_{l=0}^{j-1} \rho_{i_l} \sum_{n=1}^{N} \tilde{\theta}_{\tilde{s}(i_l)-n}\right)\right].\end{aligned}$$

Note that

$$E\left[\sum_{n=1}^{N} \tilde{\theta}_{\tilde{s}(i_l)-n}\right] = E\left[C_{s(i_j)}\right] = E[C].$$

Then we have

$$E\left[X_{i_j}^{i_j} - X_{s(i_j)}^{i_j}\right] = \lambda_{i_j}\left[\sum_{l=\tilde{s}(i_j)}^{i-1} E[\tilde{\theta}_l] + \sum_{l=0}^{j-1}(\bar{d}_{i_l} + \rho_{i_l} E[C])\right], \tag{22}$$

where $E[\tilde{\theta}_l]$ is given by (8) and (9).

- $\mathrm{var}\left(X_{s(i_j)}^{i_j}\right) = \lambda_{i_j}^2 \ \mathrm{var}[C_{s(i_j)}] + \lambda_{i_j} E[C]$

(see e.g. [10], pp. 186–187, eqs. 5.58, 5.61).

- $E\left[X_{s(i_j)}^{i_j}\left(X_{i_j}^{i_j} - X_{s(i_j)}^{i_j}\right)\right] - E\left[X_{s(i_j)}^{i_j}\right]E\left[X_{i_j}^{i_j} - X_{s(i_j)}^{i_j}\right]$

$$= \lambda_{i_j}^2 E\left[\left(\sum_{n=1}^{N}\left(\tilde{\theta}_{\tilde{s}(i_j)-n} - E\left[\tilde{\theta}_{\tilde{s}(i_j)-n}\right]\right)\right) \times \left(\sum_{m=\tilde{s}(i_j)}^{i-1}\left(\tilde{\theta}_m - E\left[\tilde{\theta}_m\right]\right)\right)\right] \tag{23}$$

$$+ \lambda_{i_j}^2 \sum_{l=0}^{j-1} \rho_{i_l} E\left[\left(\sum_{n=1}^{N}\left(\tilde{\theta}_{\tilde{s}(i_l)-n} - E\left[\tilde{\theta}_{\tilde{s}(i_l)-n}\right]\right)\right)\right.$$

$$\left.\times\left(\sum_{m=1}^{N}\left(\tilde{\theta}_{\tilde{s}(i_j)-m} - E\left[\tilde{\theta}_{\tilde{s}(i_j)-m}\right]\right)\right)\right]. \tag{24}$$

Expression (23) is given by

$$(23) = \begin{cases} \lambda_{i_j}^2 \displaystyle\sum_{n=s(i_j)}^{i-1}\left(\sum_{m=s(i_j)+N}^{2N} r_{nm} + \sum_{m=1}^{s(i_j)-1} r_{nm}\right); & s(i_j) \le i, \\ \lambda_{i_j}^2 \displaystyle\sum_{n=s(i_j)}^{s(i_j)+N-1}\left(\sum_{m=s(i_j)+N}^{2N} r_{mn} + \sum_{m=1}^{i-1} r_{mn}\right); & s(i_j) > i. \end{cases} \tag{25}$$

To compute (24), we have to compute

$$E\left[\left(\sum_{n=1}^{N}\left(\tilde{\theta}_{\tilde{s}(i_l)-n} - E\left[\tilde{\theta}_{\tilde{s}(i_l)-n}\right]\right)\right) \times \left(\sum_{m=1}^{N}\left(\tilde{\theta}_{\tilde{s}(i_j)-m} - E\left[\tilde{\theta}_{\tilde{s}(i_j)-m}\right]\right)\right)\right] \tag{26}$$

for each i_l, $i = 1, 2, \ldots, N$, $l, j = 1, 2, \ldots, m(i)$. For $s(i_l) \le s(i_j) \le i$ (and similarly for $s(i_j) \le s(i_l) \le i$), we have

$$(26)=\sum_{n=s(i_l)}^{s(i_j)-1}\left(\sum_{m=s(i_l)+N}^{2N} r_{nm}+\sum_{m=1}^{s(i_l)-1} r_{nm}\right)$$
$$+\sum_{n=1}^{s(i_l)-1}\left(\sum_{m=s(i_l)+N}^{2N} r_{nm}+\sum_{m=1}^{n-1} r_{nm}+\sum_{m=n}^{s(i_l)-1} r_{mn}\right)$$
$$+\sum_{n=s(i_j)+N}^{2N}\left(\sum_{m=s(i_l)+N}^{n-1} r_{nm}+\sum_{m=n}^{2N} r_{mn}+\sum_{m=1}^{s(i_l)-1} r_{mn}\right).$$

For the case where $s(i_j)$, $s(i_l)\geq i$ and $s(i_l)\leq s(i_j)$ (and similarly for $s(i_l)\geq s(i_j)$), we obtain

$$(26)=\sum_{n=s(i_l)+N}^{s(i_j)+N-1}\sum_{m=s(i_l)}^{s(i_l)+N-1} r_{nm}+\sum_{n=s(i_j)}^{s(i_l)+N-1}\left(\sum_{m=s(i_l)}^{n-1} r_{nm}+\sum_{m=n}^{s(i_l)+N-1} r_{mn}\right).$$

Finally, for the case where $s(i_j)>i$ and $s(i_l)<i$ (and similarly for the case where $s(i_l)>i$ and $s(i_j)<i$), we obtain

$$(26)=\sum_{n=s(i_j)}^{s(i_l)+N-1}\left(\sum_{m=s(i_l)+N}^{2N} r_{mn}+\sum_{m=1}^{s(i_l)-1} r_{mn}\right)$$
$$+\sum_{n=s(i_l)+N}^{s(i_j)+N-1}\left(\sum_{m=s(i_l)+N}^{n-1} r_{nm}+\sum_{m=n}^{2N} r_{mn}+\sum_{m=1}^{s(i_l)-1} r_{mn}\right).$$

Auxiliary problem: We calculate the distribution of the number of jobs that arrive at station i_j during T_{i_l} – the time that station i_l is served. We denote this number by $Y_{i(j,l)}$. The jobs that are served at station i_l are those that arrived at the station during $C_{s(i_l)}$. We denote them by n. The moment generating function of $Y_{i(j,l)}$ satisfies:

$$Y^*_{i(j,l)}(z)=T^*_{i_l}(\lambda_{i_j}-\lambda_{i_j}z)=E\exp\{-\lambda_{i_j}(1-z)T_{i_l}\}=E\left[E\exp\{-\lambda_{i_j}(1-z)T_{i_l}\mid n\}\right]$$

$$=E\left[\left(B^*_{i_l}(\lambda_{i_j}-\lambda_{i_j}z)\right)^n\right]=E\left\{E\left[\left(B^*_{i_l}(\lambda_{i_j}-\lambda_{i_j}z)\right)^n\mid\tilde{\vartheta}_{i-1}\right]\right\}$$

$$=E\exp\left\{-\lambda_{i_l}\left(1-B^*_{i_l}(\lambda_{i_j}-\lambda_{i_j}z)\right)\sum_{m=1}^{N}\tilde{\theta}_{\bar{s}(i_l)-m}\right\}.$$

6. A pseudo conservation law

6.1. COMPUTING THE CONSERVATION LAW

In Boxma and Groenendijk [5], a "pseudo" conservation law has been derived for polling systems with mixed service strategies. The derivation of this law is based on a work decomposition theorem. Recently, Boxma [6] extended this theorem to much more general single-server systems with non-serving intervals. Here, we adapt *theorem 1* of Boxma and Groenendijk [5] to our network with the synchronized gated discipline.

The derivation of a "pseudo" conservation law for our network proceeds in the same way as in Boxma and Groenendijk [5], and we may obtain an equivalent equation to eq. (3.10) in [5] with the appropriate modifications in notation,

$$\sum_{i=1}^{N}\sum_{j=0}^{m(i)} \rho_{i_j} E[W_{i_j}] = \rho \frac{\sum_{i=1}^{N}\sum_{j=0}^{m(i)} \lambda_{i_j} b_{i_j}^{(2)}}{2(1-\rho)} + \rho \frac{d^{(2)}}{2\bar{d}} + \frac{\bar{d}}{2(1-\rho)}\left[\rho^2 - \sum_{i=1}^{N}\sum_{j=0}^{m(i)} \rho_{i_j}^2\right] + \sum_{i=1}^{N}\sum_{j=0}^{m(i)} E\left[M_{i_j}^{(1)}\right], \tag{27}$$

where $M_{i_j}^{(1)}$, $i = 1, 2, \ldots, N$, $j = 0, 1, \ldots, m(i)$, is defined as in [5] and is equal to the amount of work in station i_j at a departure epoch of the server from station i_j.

The meaning of the terms on the right-hand side of (27) is given in [5] immediately after eq. (3.10). It turns out that $E[M_{i_j}^{(1)}]$ is the only term on the right-hand side of (27) that depends on the service strategy at station i_j, as expected. The quantity $E[M_{i_j}^{(1)}]$ represents the mean amount of work that arrived at station i_j during the move of the server from station $s(i_j)$ to station i_j including the service times of these stations and the stations between them, and the overall walking times from station $s(i_j)$ to station i_j. $E[M_{i_j}^{(1)}]$ *depends on the service strategy at station* i_j. We now turn to the determination of $E[M_{i_j}^{(1)}]$ in the case of synchronized gated discipline.

For $i = 1, 2, \ldots, N$, we have the following:

$$E\left[M_i^{(1)}\right] = \rho_i^2 E[C] = \rho_i^2 \frac{\bar{d}}{1-\rho}. \tag{28}$$

For the synchronized stations, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, m(i)$, we have the following:

$$E\left[M_{i_j}^{(1)}\right] = \rho_{i_j}^2 E[C] + E\left[X_{i_j}^{i_j} - X_{s(i_j)}^{i_j}\right]\bar{b}_{i_j} = \rho_{i_j}^2 \frac{\bar{d}}{1-\rho} + E\left[X_{i_j}^{i_j} - X_{s(i_j)}^{i_j}\right]\bar{b}_{i_j}. \tag{29}$$

Using (8) and (9) in (22), we obtain

$$E\left[X_{i_j}^{i_j} - X_{s(i_j)}^{i_j}\right] = \lambda_{i_j}\left[\sum_{l=\bar{s}(i_j)}^{i-1} (\bar{\rho}_l E[C] + \bar{d}_{l_f}) + \sum_{l=0}^{j-1} (\rho_{i_l} E[C] + \bar{d}_{i_l})\right], \quad (30)$$

where in (30) we used the following definitions:

$$\bar{\rho}_{l-N} \stackrel{\text{def}}{=} \bar{\rho}_l \quad \text{and} \quad \bar{d}_{(l-N)_f} \stackrel{\text{def}}{=} \bar{d}_{l_f}, \qquad l = 1, 2, \ldots, N.$$

Substituting (28), (29) and (30) into (27), we obtain a "pseudo" conservation law for our network:

$$\sum_{i=1}^{N} \sum_{j=0}^{m(i)} \rho_{i_j} E\left[W_{i_j}\right] = \rho \frac{\sum_{i=1}^{N} \sum_{j=0}^{m(i)} \lambda_{i_j} b_{i_j}^{(2)}}{2(1-\rho)} + \rho \frac{d^{(2)}}{2\bar{d}} + \frac{\bar{d}}{2(1-\rho)}\left[\rho^2 + \sum_{i=1}^{N} \sum_{j=0}^{m(i)} \rho_{i_j}^2\right]$$

$$+ \sum_{i=1}^{N} \sum_{j=0}^{m(i)} \rho_{i_j}\left[\sum_{l=\bar{s}(i_j)}^{i-1} \left(\frac{\bar{d}}{1-\rho} \bar{\rho}_l + \bar{d}_{l_f}\right) + \sum_{l=0}^{j-1} \left(\frac{\bar{d}}{1-\rho} \rho_{i_l} + \bar{d}_{i_l}\right)\right]. \quad (31)$$

The last term on the right-hand side of (31) reflects the effect of the presence of synchronization. The weighted sum of the mean waiting times *increases* in comparison with the standard gated discipline. Without synchronization this term is omitted, and we obtain the "pseudo" conservation law for the standard gated discipline.

6.2. OPTIMIZING THE TOPOLOGY

Assume that we are free to choose the topology of the polling system; more precisely, we assume that we may change the position of the different stations on the cyclic path of the server. We assume that the walking time d_{i_j} depends only on station i_j from which it starts. Our goal is to minimize the weighted sum of the mean waiting times on the left-hand side of the "pseudo" conservation law (eq. (31)).

THEOREM 5

An optimal topology for the order of different stations is obtained by placing for each father station i, $i = 1, 2, \ldots, N$, all its children i_j, $j = 0, 1, \ldots, m(i)$, between it and the subsequent father station in the cycle $(i+1)$, in increasing order of the quantity d_{i_j}/ρ_{i_j}.

Proof

We first note that for any topology, the weighted sum of the mean waiting times may be reduced if for each father station all its children are positioned between him and the subsequent father station in the cycle. In that case, from eq. (31) we obtain:

$$\sum_{i=1}^{N}\sum_{j=0}^{m(i)}\rho_{i_j}E\left[W_{i_j}\right]=\rho\,\frac{\sum_{i=1}^{N}\sum_{j=0}^{m(i)}\lambda_{i_j}b_{i_j}^{(2)}}{2(1-\rho)}+\rho\,\frac{d^{(2)}}{2\overline{d}}+\frac{\overline{d}}{2(1-\rho)}\left[\rho^2+\sum_{i=1}^{N}\sum_{j=0}^{m(i)}\rho_{i_j}^2\right]$$

$$+\sum_{i=1}^{N}\sum_{j=0}^{m(i)}\rho_{i_j}\left[\sum_{n=0}^{j-1}\overline{d}_{i_n}+\frac{\overline{d}}{1-\rho}\sum_{n=0}^{j-1}\rho_{i_n}\right]. \tag{32}$$

From eq. (32), we see that the weighted sum of the mean waiting times is independent of the order of the father stations. Moreover, we see that the optimization problem decomposes into N independent optimization problems; for each $1\le i\le N$, we wish to minimize

$$\sum_{j=0}^{m(i)}\rho_{i_j}\left[\sum_{n=0}^{j-1}\overline{d}_{i_n}+\frac{\overline{d}}{1-\rho}\sum_{n=0}^{j-1}\rho_{i_n}\right]=\frac{\overline{d}}{1-\rho}\sum_{j<n}\rho_{i_j}\rho_{i_n}+\sum_{j=1}^{m(i)}\rho_{i_j}\sum_{n=0}^{j-1}d_{i_n}. \tag{33}$$

However, the first sum on the right-hand side of eq. (33) is invariant under any change between the order of child stations. Thus, for each i, $i=1,2,\dots,N$, we wish to minimize

$$\sum_{j=0}^{m(i)}\rho_{i_j}\sum_{n=0}^{j-1}\overline{d}_{i_n}. \tag{34}$$

Assume any order of the stations i_j, $j=1,2,\dots,m(i)$. Consider *any* two adjacent stations, say i_l and i_n, at positions k and $k+1$ ($k=1,2,\dots,m(i)-1$), respectively, and assume that $d_{i_l}/\rho_{i_l}>d_{i_n}/\rho_{i_n}$. Using interchange arguments, we interchange the positions of stations i_l and i_n. The change in the quantity given in (34) is $\rho_{i_l}d_{i_n}-\rho_{i_n}d_{i_l}$. Then, the above order is improved (in the sense of minimizing (34)) if the positions of the stations i_l and i_n are interchanged. This completes the proof of the theorem. □

7. Numerical examples

To obtain the expected waiting times at the different stations, one must calculate the second moments of the station times. This involves solving $N(2N-1)$

equations given in (14) and (15). It is convenient to solve these equations by successive substitutions, using the *Gauss–Seidel iterative procedure.*

There are approximately $4N^3$ multiplications and $4N^3$ additions per iteration. Humblet [9] reports that this procedure does indeed converge for a similar set of equations (for the standard gated discipline).

In what follows, several numerical examples are demonstrated. In each of them, the expected waiting times in the different stations of the network are calculated and agree with the "pseudo" conservation law. Ten iterations were sufficient to obtain the second moments with an accuracy of about $10^{-6}\%$ in the expected waiting times. In all the networks plotted in the following figures, a small circle indicates a "child" station and a small box indicates a "father" station. An arrow is drawn from a "father" station to its child.

Example I

The purpose of this example is to demonstrate the solution of a general polling system with several precedence constraints, and to compare its performance with the standard gated discipline and the "optimal topology" version (see section 6).

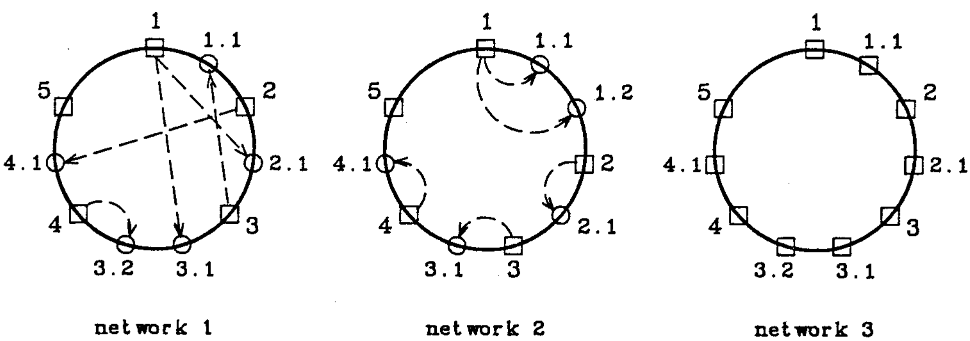

Fig. 1. Networks 1, 2 and 3.

A schematic figure of the network appears in fig. 1. It consists of 10 stations; 5 of them are "father" stations and the others are "child" stations. The network is symmetric with deterministic walking times. The following parameters are considered:

$$\bar{b}_{i_j} = 0.25, \quad b_{i_j}^{(2)} = 0.125, \quad \bar{d}_{i_j} = 1, \quad d_{i_j}^{(2)} = 1.$$

The expected waiting times in the different stations of the network and the average waiting time in the network ($\bar{W}_{net1}$) are plotted in fig. 2 as functions of the total system load ρ. All "father" stations have approximately the same expected

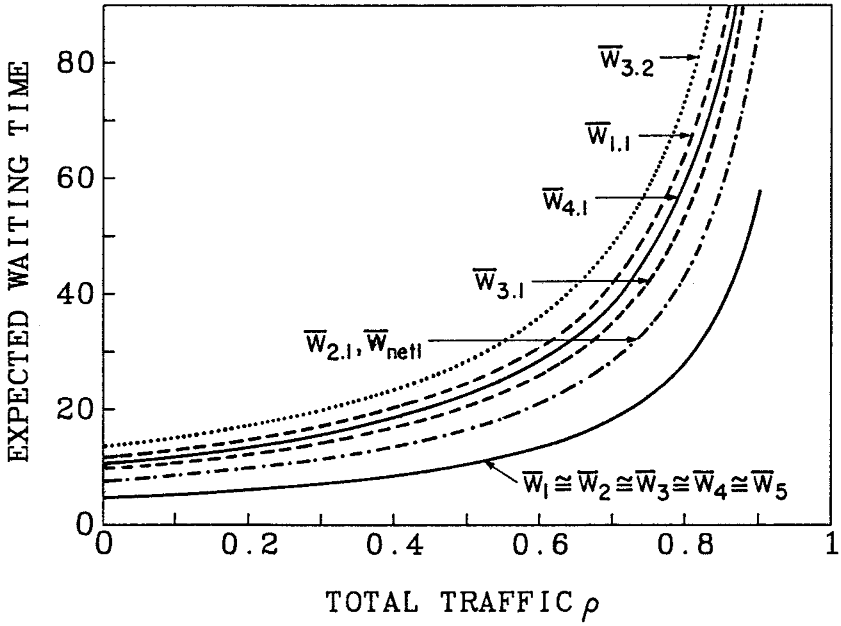

Fig. 2. Expected waiting times in the different stations of network 1, and in network 1 versus total load in network 1 ρ. ($\bar{b}_{i_j} = 0.25$, $b_{i_j}^{(2)} = 0.125$, $\bar{d}_{i_j} = 1$, $d_{i_j}^{(2)} = 1$.)

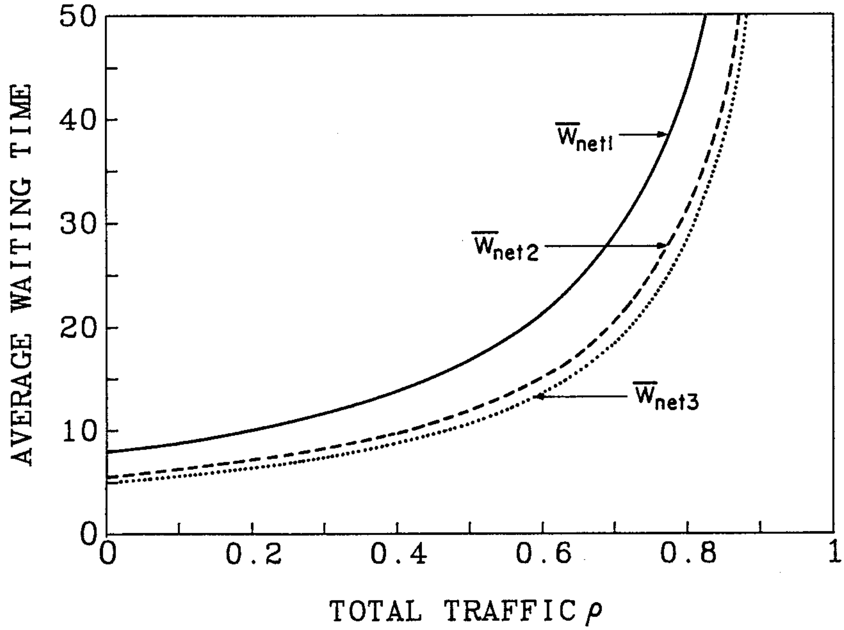

Fig. 3. Average waiting times in networks 1, 2 and 3 versus total load in the network ρ. ($\bar{b}_{i_j} = 0.25$, $b_{i_j}^{(2)} = 0.125$, $\bar{d}_{i_j} = 1$, $d_{i_j}^{(2)} = 1$.)

waiting time, and it is the smallest among the stations of the network. For a "child" station, the expected waiting time increases as the distance (number of stations in the direction of the server movement) between its "father" and it increases.

Schematic figures of networks 2 and 3 appear in fig. 1. They are symmetric networks with the same parameters as in network 1. The average waiting times in networks 1, 2 and 3 ($\bar{W}_{net1}$, $\bar{W}_{net2}$ and $\bar{W}_{net3}$) are plotted in fig. 3. Network 2 is the "optimal topology" version of network 1, as described in section 6. The average waiting time in network 2 is lower than the average waiting time in network 1, as proved in section 6. Network 3 is the "gated" version of network 1, where all the stations are "father" stations. The average waiting time in network 3 is the smallest among the three networks, as noticed in section 6.

Example II

In this example, we show the effect of the distances between a "father" station and its "child" station on the expected waiting times in the "father" stations and the "child" stations, and also on the average waiting time in the network. To

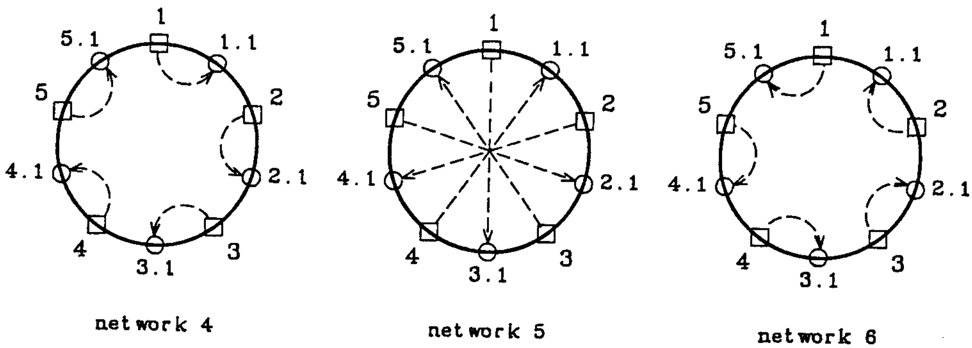

Fig. 4. Networks 4, 5 and 6.

show this, we use 3 networks as shown in fig. 4 (networks 4–6). All these networks are symmetric with the following parameters

$$\bar{b}_{i_j} = 0.25, \quad b_{i_j}^{(2)} = 1, \quad \bar{d}_{i_j} = 0.01, \quad d_{i_j}^{(2)} = 0.001.$$

In fig. 5, the expected waiting times in "father" and "child" stations and the average waiting times in network (i), $i = 4, 5, 6$ ($\bar{W}_{father}(net\ i)$, $\bar{W}_{child}(net\ i)$ and $\bar{W}_{net\ i}$) are plotted as functions of the total network load ρ. The expected waiting time in a "father" station (it is the same for all "father" stations) *decreases* as the

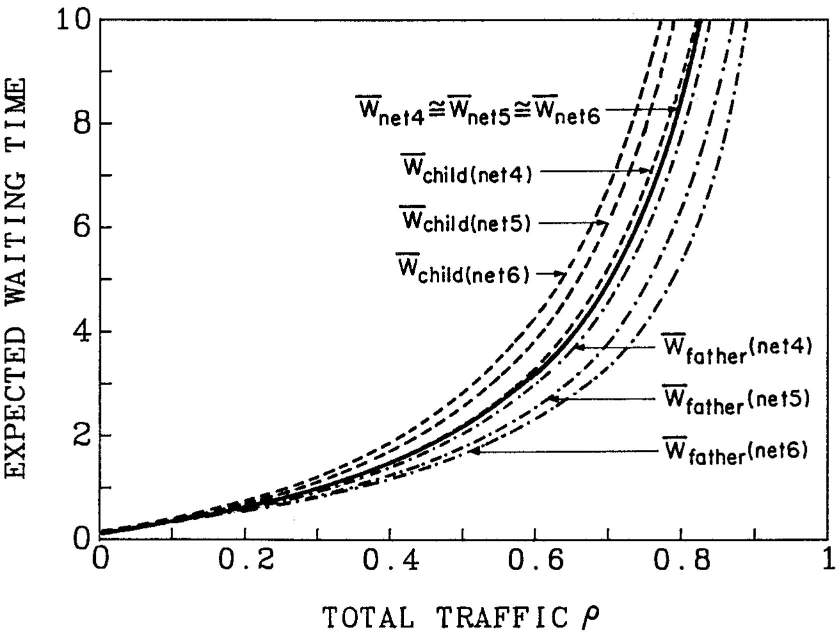

Fig. 5. Expected waiting times in "father" and "child" stations and in the network, for networks 4, 5 and 6 versus total load in the network ρ. ($\bar{b}_{i_j} = 0.25$, $b_{i_j}^{(2)} = 1$, $\bar{d}_{i_j} = 0.01$, $d_{i_j}^{(2)} = 0.001$.)

distance between "father" and "child" stations increases. For a "child" station, opposite behavior is observed. The average waiting time in the network remains approximately unchanged as the distance increases; this can be explained by the fact that $\bar{d}$ and $d^{(2)}$ are small, and in that case the "pseudo" conservation law becomes a conservation law. We use the following parameters to illustrate the behavior of the network for larger $\bar{d}$ and $d^{(2)}$:

$$\bar{b}_{i_j} = 0.25, \quad b_{i_j}^{(2)} = 0.125, \quad \bar{d}_{i_j} = 1, \quad d_{i_j}^{(2)} = 1.$$

The expected waiting times in "father" and "child" stations are plotted in fig. 6 as functions of the total network load ρ, for networks 4, 5 and 6. The expected waiting time for a "father" station is approximately the same for the three networks, and for a "child" station it increases as the distance increases. In fig. 7, we plot the average waiting time in the network, for networks 4, 5 and 6. It can be seen from fig. 7 that the average waiting time in the network increases noticeably as the distance increases.

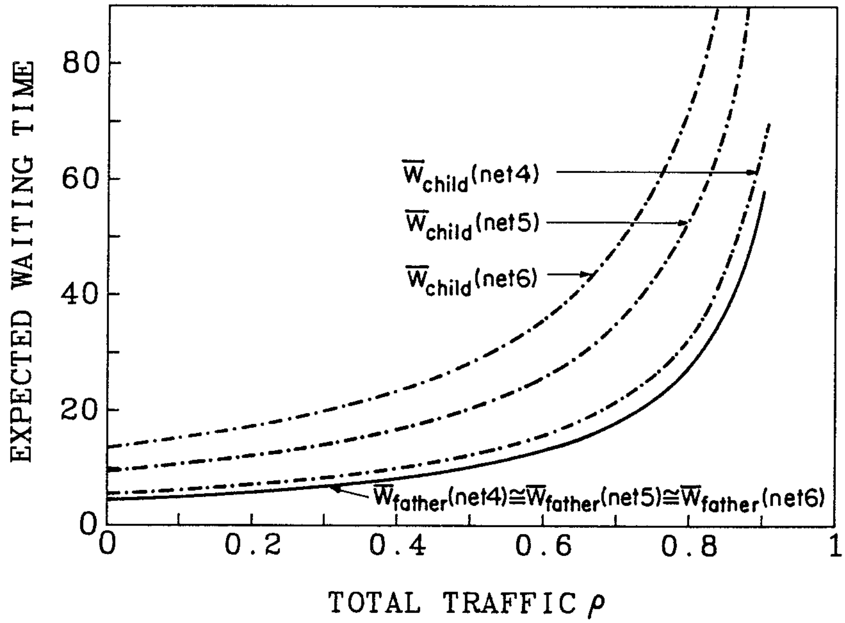

Fig. 6. Expected waiting times in "father" and "child" stations for networks 4, 5 and 6 versus total load in the network ρ. ($\bar{b}_{i_j} = 0.25$, $b_{i_j}^{(2)} = 0.125$, $\bar{d}_{i_j} = 1$, $d_{i_j}^{(2)} = 1$.)

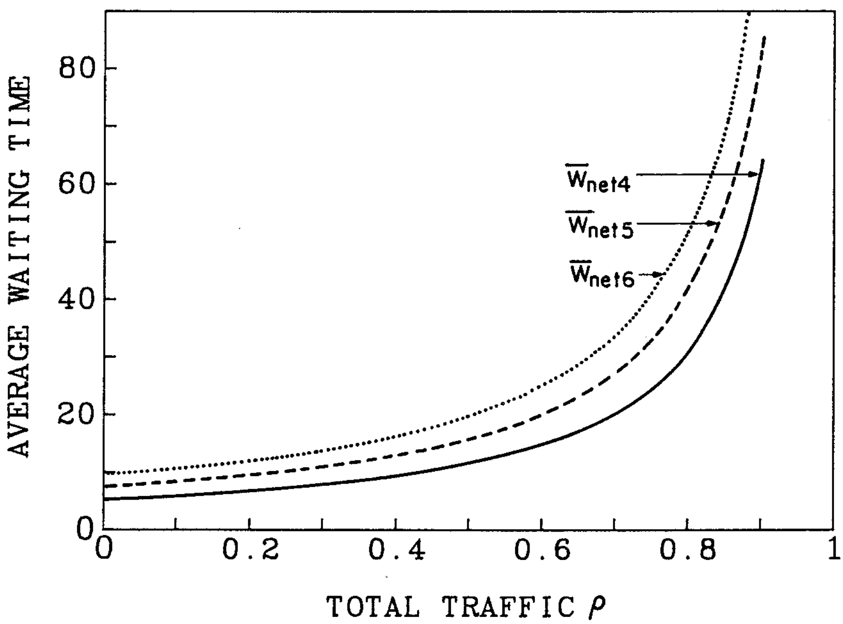

Fig. 7. Average waiting times in networks 4, 5 and 6 versus total load in the network ρ. ($\bar{b}_{i_j} = 0.25$, $b_{i_j}^{(2)} = 0.125$, $\bar{d}_{i_j} = 1$, $d_{i_j}^{(2)} = 1$.)

Example III

In this example, we demonstrate the effect of the number of synchronized stations in the network on the expected waiting times in the "father" and "child" stations and also on the average waiting time in the network. As extreme cases of our analysis, we have the following two service disciplines: In the one extreme, there is the standard gated scheme (without synchronized stations at all) and in the other extreme, there is the globally gated scheme in which all the stations except one (the "father" station) are synchronized stations.

In [7], it was reported that for a symmetric, globally gated network in which $d = 0$, stations positioned at the "first half circle" prefer a globally gated regime (in the sense that lower expected waiting times are achieved), while the others

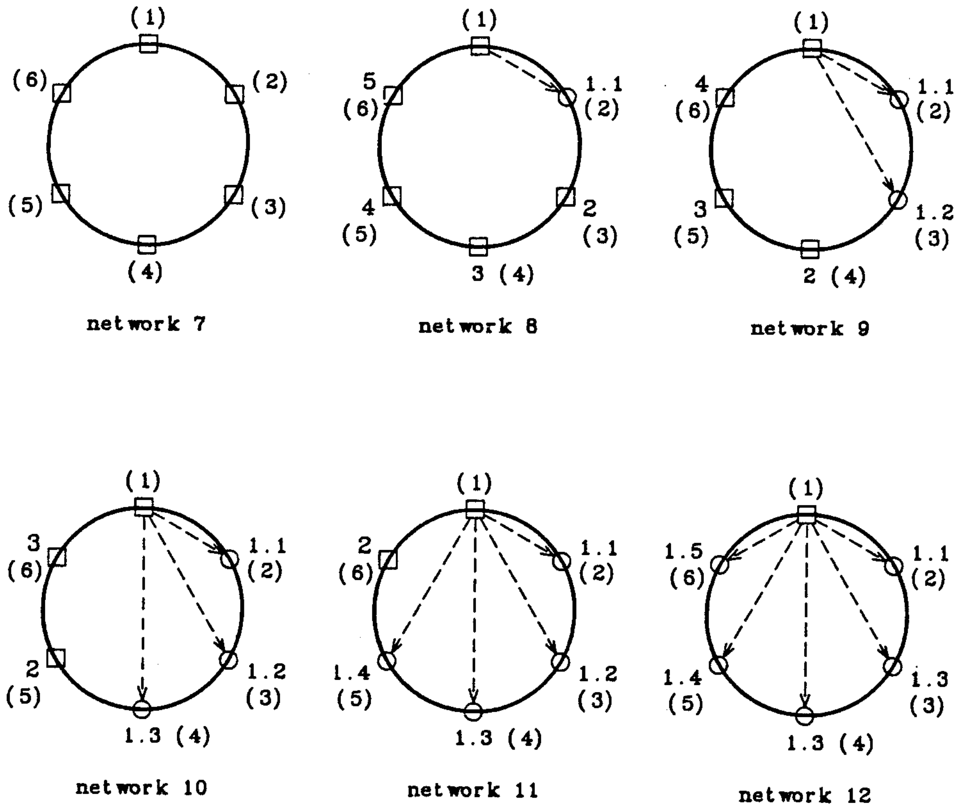

Fig. 8. Networks 7–12.

prefer the standard gated scheme. In order to investigate the behavior of polling systems with synchronization constraints for small values of d, we use 6 networks as demonstrated in fig. 8 (networks 7–12). The station numbers that appear in brackets indicate the station numbers used in the following figures (figs. 9–12).

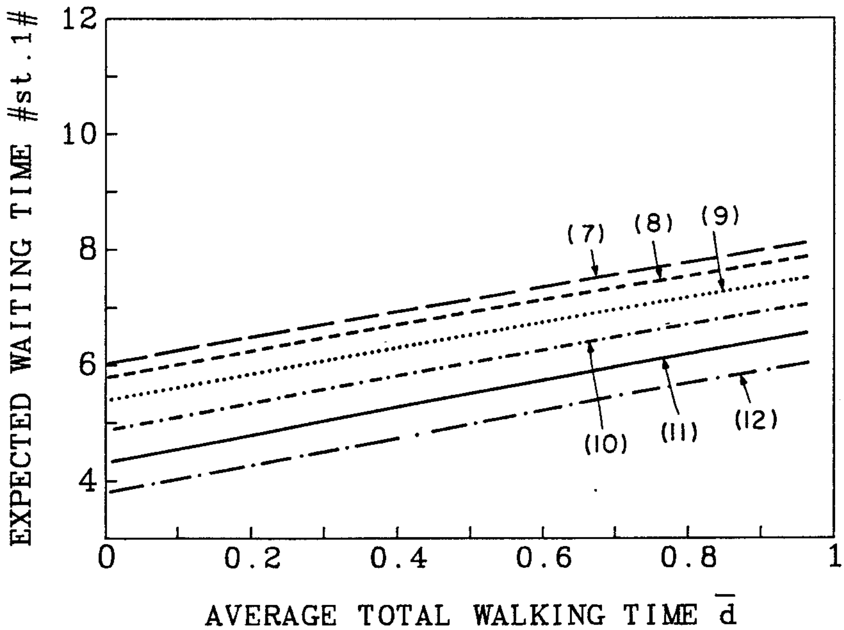

Fig. 9. Expected waiting time in station (1) versus average total walking time in the network $\bar{d}$. ($\bar{b}_{i_j} = 0.25$, $b_{i_j}^{(2)} = 1$, $\rho = 0.75$, $d_{i_j}^{(2)} = \bar{d}_{i_j}^2$.)

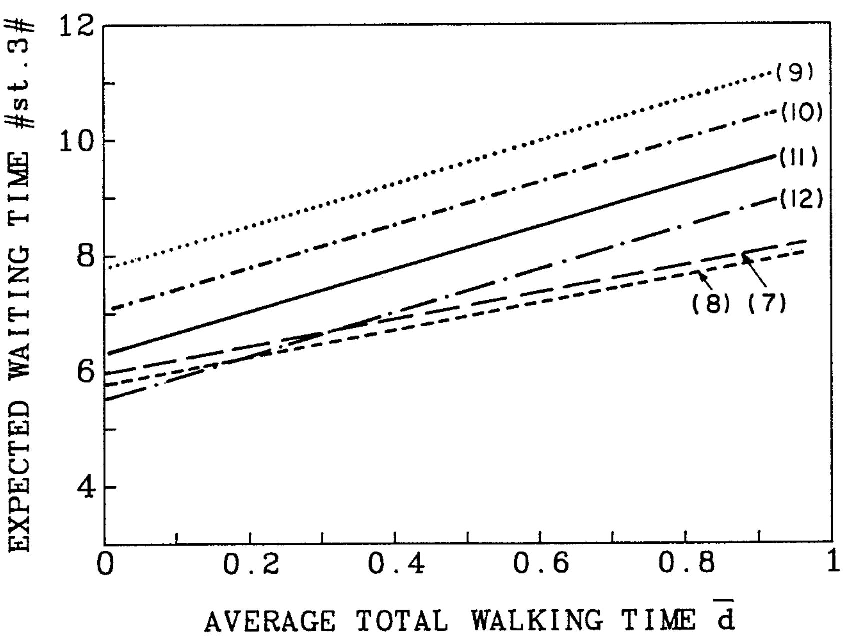

Fig. 10. Expected waiting time in station (3) versus average total walking time in the network $\bar{d}$. ($\bar{b}_{i_j} = 0.25$, $b_{i_j}^{(2)} = 1$, $\rho = 0.75$, $d_{i_j}^{(2)} = \bar{d}_{i_j}^2$.)

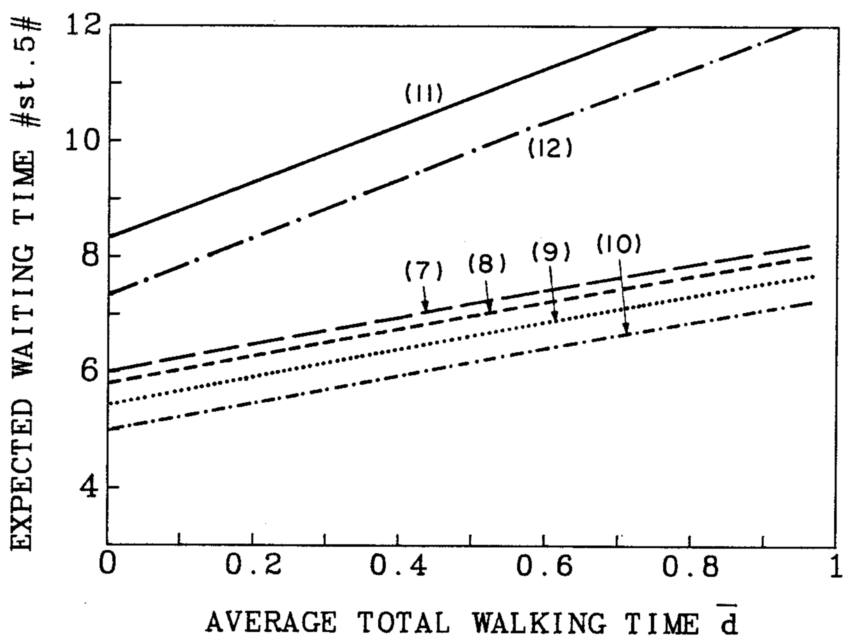

Fig. 11. Expected waiting time in station (5) versus average total walking time in the network $\bar{d}$. ($\bar{b}_{i_j} = 0.25$, $b_{i_j}^{(2)} = 1$, $\rho = 0.75$, $d_{i_j}^{(2)} = \bar{d}_{i_j}^2$.)

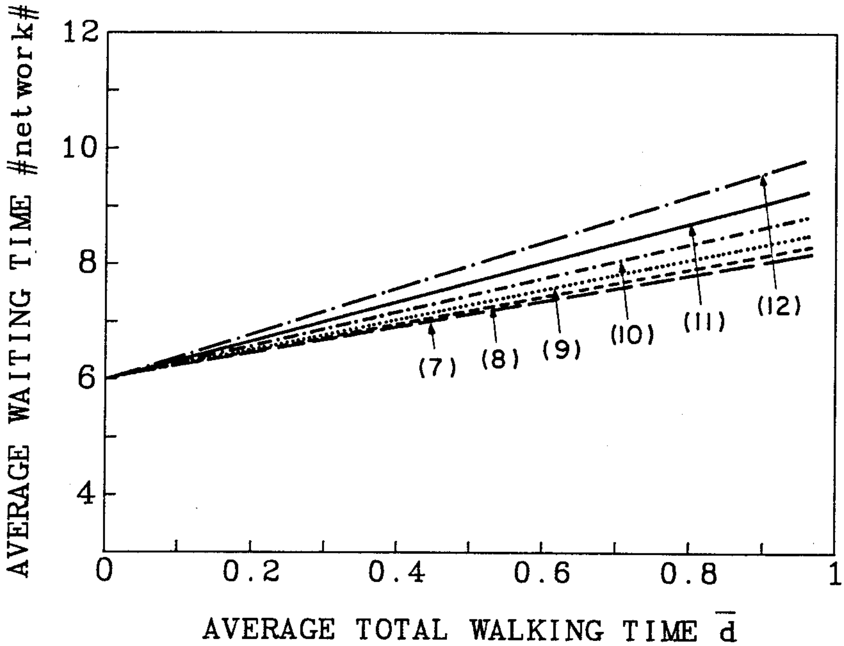

Fig. 12. Average waiting time in the network versus average total walking time in the network $\bar{d}$. ($\bar{b}_{i_j} = 0.25$, $b_{i_j}^{(2)} = 1$, $\rho = 0.75$, $d_{i_j}^{(2)} = \bar{d}_{i_j}^2$.)

In each of the networks there are 6 stations, and the only station with any children is station 1. Network 7 is the standard gated version and network 12 is the globally gated version. The other networks are versions that lie between the globally gated and the standard gated schemes. For all networks, we use the following parameters:

$$\bar{b}_{i_j} = 0.25, \quad b_{i_j}^{(2)} = 1, \quad d_{i_j}^{(2)} = \bar{d}_{i_j}^2, \quad \rho = 0.75.$$

That is, $\mathrm{var}(d_{i_j}) = 0$.

In figs. 9, 10, 11 and 12, the expected waiting times in stations 1, 3, 5 (the stations enclosed in brackets in fig. 8) and in the network, respectively, are plotted as functions of the average total walking time in the network $\bar{d}$. For each graph in figs. 9–12, we print the number of the network in which the graph was obtained near the graph (in brackets).

From fig. 9, the expected waiting time in station 1 decreases when moving from network 7 to network 12, for all values of $\bar{d}$. That is, station 1 prefers more synchronized stations in the network. From fig. 10, station 3, which is positioned in the "first half circle", prefers network 8 over the other networks for values of $\bar{d}$ greater than 0.18, prefers the globally gated regime over the other networks for values of $\bar{d}$ smaller than 0.18, and prefers the globally gated regime over the gated scheme for values of $\bar{d}$ smaller than 0.32. From fig. 11, station 5, which is positioned in the "second half circle", does not prefer the globally gated regime for any value of $\bar{d}$, and prefers network 10 (the last network before it becomes synchronized) for all values of $\bar{d}$. From fig. 12, the average waiting time in the network *increases* when moving from network 7 to network 12 for all values of $\bar{d} > 0$, that is, the average waiting time in the network increases as the number of synchronized stations increases. Note that, for $d \to 0$, the "pseudo" conservation law becomes a conservation law and the average waiting time in the network does not depend on the service strategy, and it is the same for all the networks.

8. Concluding remarks

We present in this section some further applications and possible extensions of our results, and compare the performance of our model with the corresponding standard gated discipline.

8.1. COMPARISON OF THE PERFORMANCE

By the corresponding standard gated discipline model, we mean the system which is obtained by using the same number and order of stations and the same statistics of arrivals, service times of jobs and walking times, but replacing the synchronized gating by the standard gating discipline. We therefore keep the notation

$i = 1, \ldots, N$ and $i_j, j = 1, \ldots, m(i)$ also when referring to the standard gated discipline without any longer assuming that stations i_j are synchronized to other stations.

As we saw in subsection 4.1 (eq. (7)), the expectation of the cycle duration remains the same. Another quantity that remains unchanged is $E[X_i^i]$: the expected number of jobs in a father station i, $i = 1, \ldots, N$, seen at the polling instants to that station. This can be seen in eq. (19) (which holds also for $j = 0$). However, the synchronization mechanism causes an increase in these quantities in all synchronized stations. $E[X^{i_j}_{s(i_j)}]$ as computed in the synchronized regime turns out to be equal to $E[X^{i_j}_{i_j}]$ in the corresponding standard gated regime. However, $E[X^{i_j}_{i_j}]$ as computed in the synchronized regime, is strictly larger for any $j = 1, \ldots, m(i)$ than the same quantity in the corresponding standard gated regime. The difference is equal to the right-hand side of eq. (22). Finally, as we saw in section 6 (eq. (31)), the weighted sum of the mean waiting times increases when adding synchronization; the difference is given in the last term of eq. (31).

In [7], the individual expected waiting times for the symmetric gated scheme and for the globally gated scheme were compared. In the case of a polling system with general synchronization constraints, an explicit comparison is impossible, because no explicit expressions exist for the expected waiting times. Instead, they are expressed as functions of the second moments which are given implicitly by a set of $N(2N-1)$ equations, and poses a different structure from the equivalent set of equations in the standard gated scheme.

We carried out the calculation of the waiting times distributions and moments through an extension of the "station time" approach. The alternative "buffer occupancy" approach (as can be found in [15]), i.e. the direct calculation of the distributions of the quantities X_k^l (where k and l stand for arbitrary stations) was not suitable for our synchronized model; the problem of using that method directly is that it becomes extremely complex to appropriately represent information needed from more than one cycle back in a way that yields recursive equations for calculating the moment generating functions.

8.2. THE EXHAUSTIVE SYNCHRONIZED DISCIPLINE

Assume the same partition between father and child stations as in the gated synchronized model, and assume that the father stations are served by an exhaustive discipline. We require that the precedence constraints are to hold between the father and its child stations, i.e. jobs in any child station can be processed only if all jobs that arrived previously at the father station have been served. In order to meet these precedence constraints, we may use the following synchronization discipline: at the *end* of the service of every father station, gates are closed in all its child stations. When the server arrives at a child station, it serves only the jobs that arrived there before the gate was closed. The same methodology that we developed in this paper can be applied for analyzing this new discipline.

8.3. NESTED PRECEDENCE CONSTRAINTS

In parallel computations, we often encounter precedence constraints which are much more complex than the ones we analyze; these are described either by trees of arbitrary depth or by more general precendence graphs (allowing cyclic dependence between jobs) (see e.g. [2, 3] and references therein). We could consider a nested precedence order between stations for creating gates, e.g. when station 1 is polled it closes a gate in station 4, and when station 4 is polled it closes a gate in station 7. However, it can easily be seen that such a discipline fails to meet with the original nested precedence constraints. For example, a problem arises in station 7, where jobs that arrived after the polling instant at station 1 will be served before the jobs that arrived at station 4 after the polling instant at station 1. The same kind of problem arises when using variants of the exhaustive synchronized discipline. Therefore, in order to fulfill nested precedence constraints one should devise more complex service discipline.

8.4. PRIORITY POLLING SYSTEMS

Consider a single server, cyclic system with N stations, and nonzero switch-over times between the stations. Jobs from $m(i) + 1$ different classes i_j, $j = 0, 1, \dots, m(i)$, arrive at each station i, $i = 1, 2, \dots, N$ according to independent Poisson processes with intensity λ_{i_j} for class i_j jobs. Assume that service at each station i is given according to pre-assigned priorities to the different classes, where the priority of class i_j, $j = 0, 1, \dots, m(i)$, jobs increases as the index j decreases. Assume that the service strategy is the standard gated strategy, where at each polling epoch to station i a gate is closed at that station and jobs are serviced according to their priorities. Assume also that the server takes a vacation each time it finishes service of a priority class at station i. The polling system with priorities and vacations between the service of priority classes we have just described turns out to be a special case of the synchronized polling system described in this paper, where we have N father stations and for each father station, all of its children are situated between him and the following father station as described in the model description (the same topology as for the "optimal topology" described in section 6).

8.5. REDUCING THE COMPLEXITY

Actually, the evolution of the system can be described in terms of $N + Dist$ station times in place of $2N - 1$ station times described in section 3 (eq. (1)), where $Dist$ equals the largest distance (number of father stations in the direction of the server movement) between a father station and one of its child stations. Then, $0 \le Dist \le N - 1$. $Dist = 0$ when all fathers have their children situated between them and the subsequent father stations, and $Dist = N - 1$ when for some child station i_j, its father is the station $i + 1$. This fact may be used to reduce the number and complexity of the equations for the second moments. Note that for the case without synchronization, $Dist = 0$, and the state can be described by N station times as in the standard gated discipline. For the globally gated discipline, $Dist = 0$ and $N = 1$, the state can be described by one station time. We avoid the use of $Dist$ to simplify notation.

Appendix A: Proof of theorem 4

To obtain the second moments, we differentiate (5) twice, with respect to the different x_j's:

$$\frac{\partial^2 \tilde{G}_i(\boldsymbol{x})}{\partial x_0^2} = \frac{\partial^2}{\partial x_0^2}\left[D_i^*(x_0)\tilde{G}_{i-1}(\boldsymbol{\alpha})\right]$$

$$= \frac{\mathrm{d}^2 D_i^*(x_0)}{\mathrm{d}x_0^2}\tilde{G}_{i-1}(\boldsymbol{\alpha}) + 2\frac{\mathrm{d}D_i^*(x_0)}{\mathrm{d}x_0}\sum_{n=0}^{2N-2}\frac{\mathrm{d}\tilde{\alpha}_n}{\mathrm{d}x_0}\frac{\partial \tilde{G}_{i-1}(\tilde{\boldsymbol{\alpha}})}{\partial \tilde{\alpha}_n}$$

$$+ D_i^*(x_0)\sum_{n=0}^{2N-2}\frac{\mathrm{d}^2\tilde{\alpha}_n}{\mathrm{d}x_0^2}\frac{\partial \tilde{G}_{i-1}(\tilde{\boldsymbol{\alpha}})}{\partial \tilde{\alpha}_n}$$

$$+ D_i^*(x_0)\sum_{n=0}^{2N-2}\sum_{m=0}^{2N-2}\frac{\mathrm{d}\tilde{\alpha}_n}{\mathrm{d}x_0}\frac{\mathrm{d}\tilde{\alpha}_m}{\mathrm{d}x_0}\frac{\partial^2 \tilde{G}_{i-1}(\tilde{\boldsymbol{\alpha}})}{\partial \tilde{\alpha}_n \partial \tilde{\alpha}_m}. \tag{35}$$

Denote by $\boldsymbol{0}$ the vector with $2N-1$ zeros. Let $i = 1, \ldots, N$ and $n, m = 0, 1, \ldots, 2N-2$. We have:

$$\left.\frac{\mathrm{d}D_i^*(x_0)}{\mathrm{d}x_0}\right|_{\boldsymbol{x}=\boldsymbol{0}} = -\bar{d}_{i_f}, \qquad \left.\frac{\mathrm{d}^2 D_i^*(x_0)}{\mathrm{d}x_0^2}\right|_{\boldsymbol{x}=\boldsymbol{0}} = d_{i_f}^{(2)},$$

$$\left.\frac{\mathrm{d}\tilde{\alpha}_n}{\mathrm{d}x_0}\right|_{\boldsymbol{x}=\boldsymbol{0}} = \sum_{j=0}^{m(i)}\tilde{I}^i(j,n)\rho_{i_j}, \qquad \left.\frac{\mathrm{d}^2\tilde{\alpha}_n}{\mathrm{d}x_0^2}\right|_{\boldsymbol{x}=\boldsymbol{0}} = -\sum_{j=0}^{m(i)}\tilde{I}^i(j,n)\lambda_{i_j}b_{i_j}^{(2)},$$

$$\left.\frac{\partial \tilde{G}_{i-1}(\tilde{\boldsymbol{\alpha}})}{\partial \tilde{\alpha}_n}\right|_{\boldsymbol{x}=\boldsymbol{0}} = -E[\tilde{\theta}_{i-n-1}], \qquad \left.\frac{\partial^2 \tilde{G}_{i-1}(\tilde{\boldsymbol{\alpha}})}{\partial \tilde{\alpha}_n \partial \tilde{\alpha}_m}\right|_{\boldsymbol{x}=\boldsymbol{0}} = E[\tilde{\theta}_{i-n-1}\tilde{\theta}_{i-m-1}].$$

We thus obtain by computing (35) at $\boldsymbol{x} = \boldsymbol{0}$:

$$E\left[\widetilde{\theta}_i^2\right] = d_{i_f}^{(2)} + \sum_{n=0}^{2N-2}\left\{\sum_{j=0}^{m(i)} \widetilde{I}^i(j,n)\lambda_{i_j}\left(2\overline{d}_{i_f}\overline{b}_{i_j} + b_{i_j}^{(2)}\right)\right\}E\left[\widetilde{\theta}_{i-n-1}\right]$$

$$+\sum_{n=0}^{2N-2}\sum_{m=0}^{2N-2}\left\{\sum_{l=0}^{m(i)}\sum_{j=0}^{m(i)} \widetilde{I}^i(l,n)\widetilde{I}^i(j,m)\rho_{i_l}\,\rho_{i_j}\right\}E\left[\widetilde{\theta}_{i-n-1}\widetilde{\theta}_{i-m-1}\right]. \quad (36)$$

Note that

$$E\left[\widetilde{\theta}_i\right] = \sum_{n=0}^{2N-2}\left\{\sum_{j=0}^{m(i)} \widetilde{I}^i(j,n)\rho_{i_j}\right\}E\left[\widetilde{\theta}_{i-n-1}\right] + \overline{d}_{i_f}. \quad (37)$$

We thus obtain

$$\left(E\left[\widetilde{\theta}_i\right]\right)^2 = \sum_{n=0}^{2N-2}\sum_{m=0}^{2N-2}\left\{\sum_{l=0}^{m(i)}\sum_{j=0}^{m(i)} \widetilde{I}^i(l,n)\widetilde{I}^i(j,m)\rho_{i_l}\,\rho_{i_j}\right\}E\left[\widetilde{\theta}_{i-n-1}\right]E\left[\widetilde{\theta}_{i-m-1}\right]$$

$$+2\,\overline{d}_{i_f}\sum_{n=0}^{2N-2}\left(\sum_{j=0}^{m(i)} \widetilde{I}^i(j,n)\rho_{i_j}\right)E\left[\widetilde{\theta}_{i-n-1}\right] + (\overline{d}_{i_f})^2. \quad (38)$$

Similarly, we obtain by computing $\partial^2\widetilde{G}_i(\boldsymbol{x})/\partial x_0 \partial x_l$ at $\boldsymbol{x} = \boldsymbol{0}$:

$$E\left[\widetilde{\theta}_i\widetilde{\theta}_{i-l}\right] = \overline{d}_{i_f}E\left[\widetilde{\theta}_{i-l}\right] + \sum_{n=0}^{2N-2}\left(\sum_{j=0}^{m(i)} \widetilde{I}^i(j,n)\rho_{i_j}\right)E\left[\widetilde{\theta}_{i-n-1}\widetilde{\theta}_{i-l}\right], \quad (39)$$

$$l = 1, \dots, 2N-2.$$

Note that

$$E\left[\widetilde{\theta}_i\right]E\left[\widetilde{\theta}_{i-l}\right] = \left\{\sum_{n=0}^{2N-2}\left(\sum_{j=0}^{m(i)} \widetilde{I}^i(j,n)\rho_{i_j}\right)E\left[\widetilde{\theta}_{i-n-1}\right] + \overline{d}_{i_f}\right\}E\left[\widetilde{\theta}_{i-l}\right]. \quad (40)$$

From (12), (36), (38), (39) and (40), we obtain (14) and

$$r_{ii} = \operatorname{var}(d_{i_f}) + \sum_{n=0}^{2N-2}\left(\sum_{l=0}^{m(i)} \widetilde{I}^i(l,n)\lambda_{i_l}b_{i_l}^{(2)}\right)E\left[\widetilde{\theta}_{i-n-1}\right]$$

$$+\sum_{j=1}^{i-1}\rho_i(j)\left(\sum_{m=i+1}^{2N}\rho_i(m)r_{jm} + \sum_{m=1}^{j-1}\rho_i(m)r_{jm} + \sum_{m=j}^{i-1}\rho_i(m)r_{mj}\right)$$

$$+\sum_{j=i+1}^{2N}\rho_i(j)\left(\sum_{m=i+1}^{j-1}\rho_i(m)r_{jm} + \sum_{m=j}^{2N}\rho_i(m)r_{mj} + \sum_{m=1}^{i-1}\rho_i(m)r_{mj}\right). \quad (41)$$

Substituting (14) into (41) finally yields (15). □

Appendix B: Obtaining equation (17)

Consider the regeneration points as the time instants at which the father station $s(i_j)$ is visited and all stations are empty. A regeneration interval is the period between two consecutive regeneration points. Denote a cycle as the period between two consecutive polling instants to station $s(i_j)$. The nth cycle is denoted by C_n. Let $M \stackrel{\text{def}}{=}$ the number of cycles in a regeneration interval.

Define the following random variables:

$\sigma^{i_j}(m, n)$ (respectively, $\sigma^{s(i_j)}(m, n)$) is the service completion instant for the mth job served at station i_j ($s(i_j)$, respectively) in the nth cycle.

$L_{i_j}(m, n)$ is the number of jobs that a job departing from station i_j leaves at station i_j at time $\sigma^{i_j}(m, n)$.

$X^{i_j}_{i_j}(n)$ (respectively, $X^{i_j}_{s(i_j)}(n)$) is the number of jobs at station i_j at the polling instant to that station (to its father station $s(i_j)$, respectively) in the nth cycle.

When adding a hat to the above random variables (and to others), we shall mean that n refers to the nth cycle in a typical regeneration interval.

Due to the well-known PASTA phenomenon (Poisson Arrivals See Time Averages [13]) and the fact that the state changes by unit step values (one by one) only, $Q_{i_j}(z)$ (defined in section 5.2) also stands for the moment generating function of the number of jobs at station i_j in steady-state regime at an *arbitrary time*.

Following Takagi ([15], pp. 77–79, 109), we have

$$E\left[Q_{i_j}(z)\right] = E\left[z^{L_{i_j}}\right] = \frac{E \sum_{n=1}^{M} \left[\sum_{m=1}^{\hat{X}^{i_j}_{s(i_j)}(n)} z^{\hat{L}_{i_j}(m,n)} \right]}{\lambda_{i_j} E \sum_{n=1}^{M} \hat{C}_n},$$

where L_{i_j} was defined in section 5.2.

Since we assume that the system is stable (ergodic), it follows that

$$\lim_{t \to \infty} \frac{1}{t} \sum_{n=1}^{t} C_n = EC \quad \text{w.p.1.}$$

On the other hand, from the theory of Renewal Reward Processes (e.g. [14], p. 279) it can be seen that

$$\lim_{t \to \infty} \frac{1}{t} \sum_{n=1}^{t} C_n = \frac{E \sum_{n=1}^{M} \hat{C}_n}{EM} \quad \text{w.p.1.}$$

Combining these equations, we obtain

$$EC = \frac{E\sum_{n=1}^{M} \hat{C}_n}{EM}.$$

(In some references, such as in [15], Wald's Theorem is used improperly to obtain the same result.)

By using similar arguments, we obtain

$$E\left[\sum_{m=1}^{X_{s(ij)}^{ij}} z^{L_{ij}(m)}\right] = \frac{E\sum_{n=1}^{M}\left[\sum_{m=1}^{\hat{X}_{s(ij)}^{ij}(n)} z^{\hat{L}_{ij}(m,n)}\right]}{EM},$$

from which it follows that

$$E\left[Q_{i_j}(z)\right] = \frac{E\left[\sum_{m=1}^{X_{s(ij)}^{ij}} z^{L_{ij}(m)}\right]}{\lambda_{i_j} EC}.$$

The rest of the calculation follows as in [15].

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