Structured Priority Queueing Systems with Applications to Packet-Radio Networks *

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Received 13 August 1982
Revised 8 March 1983

In this paper we investigate a certain class of systems containing dependent discrete time queues. This class of systems consists of N nodes transmitting packets to each other or to the outside of the system over a common shared channel, and is characterized by the fact that access to the channel is assigned according to priorities that are preassigned to the nodes. To each node a given probability distribution is attached, that indicates the probabilities that a packet transmitted by the node is forwarded to one of the other nodes or to the outside of the system.

Using extensively the fact that the joint generating function of the queue lengths distribution is an analytic function in a certain domain, we obtain an expression for this joint generating function. From the latter any moment of the queue lengths and also average time delays can be obtained.

The main motivation for investigating the class of systems of this paper is its applicability to several packet-radio networks. We give two examples: The first is a certain access scheme for a network where all nodes can hear each other, and the second is a three-node tandem packet-radio network.

Keywords: Discrete-Time Queues, Queueing Networks, Priority Queueing Systems, Dependent Queues, Dependent Arrivals, Packet-Radio Networks, Tandem Networks, Random Routing, Shared Channel.

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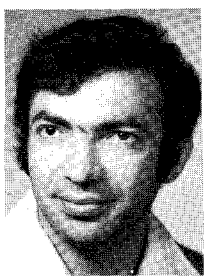

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North-Holland
Performance Evaluation 3 (1983) 265–275

1. Introduction

The purpose of the present paper is to analyze a certain class of systems containing dependent discrete-time queues. The system under consideration consists of N nodes transmitting messages to each other or to the outside of the system over a common shared channel. Fixed-length packets of data enter the system at all nodes and are buffered until the channel is made available to the node. The time is divided into slots of size corresponding to the transmission time of a packet and transmissions are started only at the beginning of a slot. The system under consideration is characterized by the fact that access to the channel is assigned according to priorities that are preassigned to the nodes. No two nodes have the same priority and a given node is allowed to transmit in a given slot only if those nodes with higher priority have empty queues. To each node we attach a given probability distribution that indicates the probabilities that a packet of data transmitted by the node is forwarded to one of the other nodes or to the outside of the network. All packets received by a node from outside or from other nodes, are buffered in a common outgoing queue. In Section 2 we formulate the system we consider, and in Section 3 we present the steady-state analysis of it and obtain the condition for steady-state and the joint generating function of the queue lengths at the nodes. From this generating function any moment of the queue lengths at the nodes can be derived and also average time delays can be obtained by using Little's law [5].

The present model is an extension to the 'loop-system' considered in [2] in two respects: in [2] nodes transmit only to the outside of the system and also the arrival processes are assumed to be independent. Using a different approach, we are able to analyze in the present paper systems with dependent inputs as well as those where nodes may transmit packets to each other. A different system where nodes send packets to each other through a loop, has been described in [7].

Out motivation for investigating the class of systems of this paper is its applicability to several packet-radio networks. In Section 4 we indicate two such applications: The first is the head of the line protocol suggested in [3] for multiplexing a small number of fully connected buffered users over a common radio channel, and the second is a three-node tandem packet-radio network.

2. Formulation

In this section we formulate the class of discrete-time queueing systems that is considered in this paper. We assume that packets arrive randomly at the N nodes of the system from N different sources, and the arrival processes may in general be correlated. Let $A_i(t)$, $i = 1, 2, \ldots, N$, $t = 0, 1, 2, \ldots,$ be the number of packets entering node i from its corresponding source in the time interval $(t, t+1]$. The input process $\{A_i(t)\}_{i=1}^N$ is assumed to be a sequence of independent and identically distributed random vectors with integer-valued elements. Let the corresponding probability distribution and generating function of the input process be

$$a(i_N, i_{N-1}, \ldots, i_1) = \text{Prob}\{A_N(t) = i_N, A_{N-1}(t) = i_{N-1}, \ldots, A_1(t) = i_1\}, \tag{1}$$

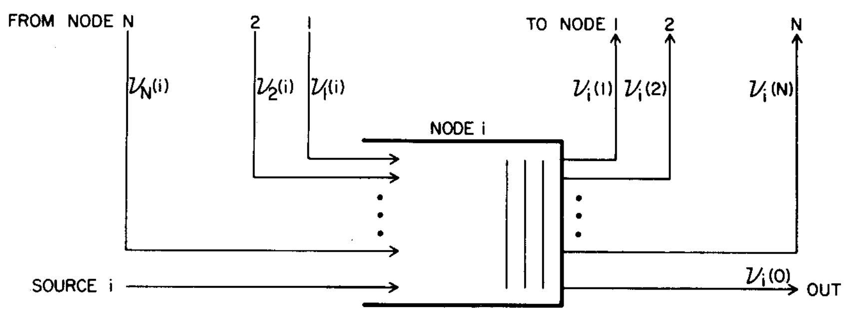

Fig. 1. An example for a node i in the network.

$$F(z)=\mathrm{E}\{z_N^{A_N(t)}\cdot z_{N-1}^{A_{N-1}(t)}\cdot \ldots \cdot z_1^{A_1(t)}\}, \tag{2}$$

where we use the notation $z=(z_N, z_{N-1},\ldots,z_1)$.

Next, we describe the departure processes from the nodes. We assume that no more than one packet may leave each node in any given time slot. Let node i have priority i, namely a packet leaves node i whenever the queues at nodes $1, 2,\ldots, i-1$ are empty and the one at node i is nonempty. In this case, node i transmits the packet at the head of its buffer to node j ($j=1, 2,\ldots, N, j\neq i$) with probability $\theta_i(j)$ or to the outside of the system with probability $\theta_i(0)$. Here we assume $\theta_i(i)=0$. The above implies that packets in the system are routed randomly through the nodes, until they eventually leave the system. It is assumed that packets indeed arrive at every node with nonzero probability and that the buffers at the nodes have infinite length. A schematic figure of a node i in the system appears in Fig. 1.

3. Steady-state distribution

To describe the evolution of the system we need several definitions. Let $L_i(t)$ be the number of packets at node i ($i=1, 2,\ldots, N$) at time t and let $U_i(L_i(t))$ ($i=1, 2,\ldots, N$) be a binary-valued random variable that takes value 1 if $L_i(t)>0$ and 0 otherwise. Also let $D_i^j(t)$, $1\leqslant i\leqslant N$, $0\leqslant j\leqslant N$ be a binary-valued random variable that takes value 1 if a packet is transmitted from i to j at time t, where $j=0$ stands for the outside of the system.

Using these definitions it is easy to see that the system under consideration evolves according to the following equations:

For $i=1, 2,\ldots, N$ and $t=0, 1, 2,\ldots,$

$$L_i(t+1)=L_i(t)+A_i(t)+\sum_{m=1}^{N} D_m^i(t)-U_i(L_i(t))\prod_{m=1}^{i-1}\left[1-U_m(L_m(t))\right]. \tag{3}$$

Consider now the steady-state joint generating function of the queue lengths distribution,

$$G(z)=\lim_{t\to\infty}\mathrm{E}\{z_N^{L_N(t)}\cdot z_{N-1}^{L_{N-1}(t)}\cdot \ldots \cdot z_1^{L_1(t)}\}. \tag{4}$$

Here we assume that the Markov chain $\{L_i(t)\}_{i=1}^{N}$ is ergodic, namely $G(z)|_{z_1=z_2=\cdots=z_N=0}>0$. For notational convenience let us define the 'routing polynoms' as follows:

$$P_i(z)=\sum_{m=1}^{N}\theta_i(m)z_m+\theta_i(0)\quad \text{for } i=1, 2,\ldots, N \quad\text{and}\quad P_{N+1}(z)\equiv 1. \tag{5}$$

Let us also define the following operators on $G(z)$:

$$\begin{aligned}
&\xi_0 G(z)\to G(z),\\
&\xi_1 G(z)\to G(z)|_{z_1=0},\\
&\xi_2 G(z)\to G(z)|_{z_1=z_2=0},\\
&\ldots,\\
&\xi_N G(z)\to G(z)|_{z_1=z_2=\cdots=z_N=0}.
\end{aligned} \tag{6}$$

We shall sometimes denote the constant $\xi_N G(z)$ by $G(\mathbf{0})$. With these notations we prove in Appendix A the following theorem.

Theorem 1. *The following holds*:

$$G(z)=F(z)\frac{\sum_{m=1}^{N}\left[z_{m+1}^{-1}P_{m+1}(z)-z_m^{-1}P_m(z)\right]\xi_m G(z)}{1-F(z)z_1^{-1}P_1(z)}\quad \textit{where } z_{N+1}\equiv 1. \tag{7}$$

In order to uniquely determine $G(z)$ we still have to determine the boundary terms $\xi_i G(z)$ for $i = 1, 2, \ldots, N$.

Determination of the constant $G(\mathbf{0})$

The constant $G(\mathbf{0})$ (or $\xi_N G(z)$) plays an important role in this system since the condition for steady-state is that $G(\mathbf{0}) > 0$.

Theorem 2. *The following holds:*

$$G(\mathbf{0}) = 1 - \sum_{l=1}^{N} \lambda_l \tag{8a}$$

where, for $1 \leqslant l \leqslant N$,

$$\lambda_l = r_l + \sum_{j=1}^{N} \lambda_j \theta_j(l) \quad \text{and} \quad r_l = \left. \frac{\partial F(z)}{\partial z_l} \right|_{z_1 = z_2 = \cdots = z_N = 1}. \tag{8b,c}$$

Proof. For $1 \leqslant i \leqslant N-1$ we define

$$G_i(\mathbf{1}) \triangleq \xi_i G(z)|_{z_{i+1} = z_{i+2} = \cdots = z_N = 1}. \tag{9}$$

If we substitute $z_i = 1$ for $i = 1, 2, \ldots, j-1, j+1, \ldots, N$ and let $z_j \to 1$ in (7), then using the normalization condition

$$G(z)|_{z_1 = z_2 = \cdots = z_N = 1} = 1, \tag{10}$$

we obtain the following set of equations:

$$1 - r_1 = [\theta_2(1) + 1] G_1(\mathbf{1}) + \sum_{m=2}^{N-1} [\theta_{m+1}(1) - \theta_m(1)] G_m(\mathbf{1}) - \theta_N(1) G(\mathbf{0}), \tag{11a}$$

$$\begin{aligned} -r_j - \theta_1(j) = & \sum_{\substack{m=1 \\ m \neq j, j-1}}^{N-1} [\theta_{m+1}(j) - \theta_m(j)] G_m(\mathbf{1}) - [1 + \theta_{j-1}(j)] G_{j-1}(\mathbf{1}) \\ & + [\theta_{j+1}(j) + 1] G_j(\mathbf{1}) - \theta_N(j) G(\mathbf{0}) \quad \text{for } j = 2, 3, \ldots, N-1, \end{aligned} \tag{11b}$$

$$-r_N - \theta_1(N) = \sum_{m=1}^{N-2} [\theta_{m+1}(N) - \theta_m(N)] G_m(\mathbf{1}) - G_{N-1}(\mathbf{1}) + G(\mathbf{0}). \tag{11c}$$

Here we have a set of N linear equations with the N constants $G_i(\mathbf{1})$, $1 \leqslant i \leqslant N-1$, and $G(\mathbf{0})$ unknown. The solution of this set of equations is:

$$G_i(\mathbf{1}) = 1 - \sum_{l=1}^{i} \lambda_l, \qquad G(\mathbf{0}) = 1 - \sum_{l=1}^{N} \lambda_l. \tag{12a,b}$$

This can be easily verified by substituting (12) into (11), thus proving the theorem. □

Determination of the boundary terms $\xi_i G(z)$, $1 \leqslant i \leqslant N-1$

The process of finding the other boundary terms $\xi_i G(z)$, $1 \leqslant i \leqslant N-1$, uses extensively the fact that the function $G(z)$ is analytic in the polydisk $|z_i| < 1$, $1 \leqslant i \leqslant N$. To proceed we shall need the following theorem.

Theorem 3. *Let* $F(z)$ *be the input generating function* (2) *and* $P_i(z)$ *the routing polynoms* (5). *For given*

$z_2, z_3, \ldots, z_N$ *with* $|z_i| < 1$, $2 \leqslant i \leqslant N$, *the following equation in* z_1,

$$F(z)P_1(z) = z_1, \tag{13}$$

has a unique solution $z_1 = z_1(z_2, z_3, \ldots, z_N)$ *in the unit circle* $|z_1| < 1$.

Proof. Here we let $|z_1| = 1$ and $|z_i| < 1$, $2 \leqslant i \leqslant N$. We distinguish between two cases: The first is the case that packets do arrive to some node l, $2 \leqslant l \leqslant N$, from its corresponding source. The second is the case that no packets arrive to nodes $2 \leqslant l \leqslant n$ from their corresponding sources. Our assumption that packets indeed arrive to all nodes implies that in the latter case, packets do arrive at node 1, and it routes some of them to at least one of the nodes l, $2 \leqslant l \leqslant N$.

Case 1. There exists some node l $(2 \leqslant l \leqslant N)$ for which the probability that a packet will arrive to it from its corresponding source is strictly positive, i.e., there exist $a(i_N, i_{N-1}, \ldots, i_1) > 0$ for some i_1 and some $i_l > 0$ $(2 \leqslant l \leqslant N)$. Therefore,

$$\begin{aligned}
&|F(z)P_1(z)| \leqslant |F(z)| = \\
&= \left| \sum_{i_N=0}^{\infty} \sum_{i_{N-1}=0}^{\infty} \cdots \sum_{i_1=0}^{\infty} a(i_N, i_{N-1}, \ldots, i_1) z_N^{i_N} \cdot z_{N-1}^{i_{N-1}} \cdot \ldots \cdot z_1^{i_1} \right| \\
&\leqslant \sum_{i_N=0}^{\infty} \sum_{i_{N-1}=0}^{\infty} \cdots \sum_{i_1=0}^{\infty} a(i_N, i_{N-1}, \ldots, i_1) |z_l^{i_l}| \\
&< \sum_{i_N=0}^{\infty} \sum_{i_{N-1}=0}^{\infty} \cdots \sum_{i_1=0}^{\infty} a(i_N, i_{N-1}, \ldots, i_1) = 1 = |z_1|.
\end{aligned} \tag{14}$$

Hence, applying Rouches' theorem [6] the claim is proved in this case.

Case 2. Packets arrive at node 1 and it routes some of them to at least one of the nodes l $(2 \leqslant l \leqslant N)$, i.e., there exist $\theta_1(l) > 0$ for some $2 \leqslant l \leqslant N$. Therefore,

$$|F(z)P_1(z)| \leqslant |P_1(z)| = |\theta_1(1)z_1 + \theta_1(2)z_2 + \cdots + \theta_1(N)z_N + \theta_1(0)| < 1 = |z_1|. \tag{15}$$

Hence, applying Rouches' theorem the proof is completed. □

We now have immediately the following corollary.

Corollary 1. *Let* $\hat{z}_1$ *denote the (unique) solution of* (13). *Let* $z^{(1)}$ *denote the vector* z *with its first component* z_1 *replaced by* $\hat{z}_1$. *Then*

$$\xi_1 G(z) = F(z^{(1)}) \frac{\sum_{m=2}^{N} \left[z_{m+1}^{-1} P_{m+1}(z^{(1)}) - z_m^{-1} P_m(z^{(1)}) \right] \xi_m G(z)}{1 - F(z^{(1)}) z_2^{-1} P_2(z^{(1)})} \tag{16}$$

This is true since $G(z)$ is analytic in the polydisk $|z_i| < 1$, $1 \leqslant i \leqslant N$. Then in this polydisk whenever the denominator of $G(z)$ vanishes, the numerator must also vanish. Since the denominator of $G(z)$ vanishes at $\hat{z}_1$ we have from (7) that

$$\sum_{m=2}^{N} \left[z_{m+1}^{-1} P_{m+1}(z^{(1)}) - z_m^{-1} P_m(z^{(1)}) \right] \xi_m G(z) + \left[z_2^{-1} P_2(z^{(1)}) - \hat{z}_1^{-1} P_1(z^{(1)}) \right] \xi_1 G(z) = 0, \tag{17}$$

which together with (13) implies (16). Notice that in (16) the boundary function $\xi_1 G(z)$ is expressed in terms of all the boundary functions $\xi_i G(z)$, $2 \leqslant i \leqslant N$. Now, exploiting the similarity between (7) and (16) we readily obtain the following corollary.

Corollary 2. *With the notation of Corollary* 1, *the equation in* z_2,

$$F(z^{(1)}) P_2(z^{(1)}) = z_2, \tag{18}$$

has a unique solution in the unit circle $|z_2| < 1$ *for given* $|z_i| < 1$, $3 \leqslant i \leqslant N$. *Let* $\hat{z}_2$ *denote this solution and* $z^{(2)}$ *denote the vector* z *with* z_1 *replaced by* $\hat{z}_1(\hat{z}_2(z_3, z_4, \ldots, z_N), z_3, \ldots, z_N)$ *and* z_2 *replaced by* $\hat{z}_2(z_3, z_4, \ldots, z_N)$. *Then*

$$\xi_2 G(z) = F(z^{(2)}) \frac{\sum_{m=3}^{N} \left[z_{m+1}^{-1} P_{m+1}(z^{(2)}) - z_m^{-1} P_m(z^{(2)}) \right] \xi_m G(z)}{1 - F(z^{(2)}) z_3^{-1} P_3(z^{(2)})}. \tag{19}$$

The proof of (19) follows the same lines as the proof of Theorem 3 and Corollary 1. In (19), $\xi_2 G(z)$ is expressed in terms of $\xi_i G(z)$, $3 \leqslant i \leqslant N$.

In general, repeating this procedure we have, for $1 \leqslant i \leqslant N-1$,

$$\xi_i G(z) = F(z^{(i)}) \frac{\sum_{m=i+1}^{N} \left[z_{m+1}^{-1} P_{m+1}(z^{(i)}) - z_m^{-1} P_m(z^{(i)}) \right] \xi_m G(z)}{1 - F(z^{(i)}) z_{i+1}^{-1} P_{i+1}(z^{(i)})}, \tag{20}$$

where $\hat{z}_1$ is defined in (13) and $\hat{z}_i$, $2 \leqslant i \leqslant N-1$, is the unique solution of

$$F(z^{(i-1)}) P_i(z^{(i-1)}) = z_i \tag{21}$$

in the unit circle $|\hat{z}_i| < 1$ for $|z_j| < 1$, $i+1 \leqslant j \leqslant N$, and $z^{(j)}$ denotes the vector z with z_i replaced by $\hat{z}_i$ for $1 \leqslant i \leqslant j$. Notice from (20) that each boundary function $\xi_i G(z)$, $1 \leqslant i \leqslant N-1$, is expressed in terms of the boundary functions $\xi_j G(z)$, $i+1 \leqslant j \leqslant N$. Specifically,

$$\xi_{N-1} G(z) = F(z^{(N-1)}) \frac{\left[1 - z_N^{-1} P_N(z^{(N-1)}) \right] G(\mathbf{0})}{1 - F(z^{(N-1)}) z_N^{-1} P_N(z^{(N-1)})}. \tag{22}$$

Since $G(\mathbf{0})$ has already been obtained in (9), we obtain $\xi_{N-1} G(z)$ from (22) and then by backward recursive substitution of $\xi_j G(z)$, $j = N-1, N-2, \ldots, 2$, in (20) we obtain $\xi_i G(z)$, $1 \leqslant i \leqslant N-2$. Thus, all the required boundary terms have been obtained, and the joint generating function $G(z)$ has been uniquely determined.

From $G(z)$ any moment of the queue lengths at the nodes can, in principle, be derived. Specifically, if we denote by $\bar{L}_i$ the average queue length at node i in steady-state, we obtain

$$\bar{L}_i = \left. \frac{\partial G(z)}{\partial z_i} \right|_{z_j = 1, 1 \leqslant j \leqslant N}. \tag{23}$$

If we assume that packets arrive at the nodes only at the end of a slot, then, using Little's law, we may also obtain the average time delays at node i denoted by T_i as follows:

$$T_i = \bar{L}_i / \lambda_i \tag{24a}$$

where λ_i—the total arrival rate at node i—is defined in (8b). The total average time delay in the system is obtained by applying Little's law to the whole system and it is given by

$$T = \sum_{i=1}^{N} \bar{L}_i \Big/ \sum_{i=1}^{N} r_i \tag{24b}$$

where r_i—the arrival rate at node i from its corresponding source— is defined in (8c).

Before proceeding we may mention that the solution we presented above can be applied to any network that belongs to the class of systems considered here, though, as will be clear from the following examples, the calculations might be very tedious.

4. Examples and applications to packet-radio networks

In this section we shall give two examples of how the results of Section 3 can be applied to packet-radio networks. Our first example would be a fully connected packet-radio network (all nodes can hear each other) that applies the channel access scheme suggested in [3]. This scheme actually allows each node in the network to know when the channel is made available to it. The second example is a three-node tandem packet-radio network where, as will be seen, the priority of the nodes is determined by the network's topology. We shall assume here that all nodes in the network share a common radio channel and that a node cannot transmit and receive a packet simultaneously.

Fully-connected packet-radio networks

By a fully-connected packet-radio network we mean that all nodes in the network can hear each other. The head-of-the-line protocol suggested in [3] for multiplexing a small number of buffered users over the common channel can be applied in such networks. Assume that priorities are preassigned to the nodes. Then, according to this protocol, each slot is divided into two parts. The first part is used to determine the node with the highest priority that has a packet ready for transmission in that slot, and the second part is used for the actual transmission of the packet at the head of the buffer at this node. In order to determine this node, the first part of the slot is divided into N equal parts (N is the number of nodes in the network) which are called minislots. A node with priority i that has a packet ready for transmission, senses the channel during minislots $1, 2, \dots, i-1$ and only if the channel is idle during this period, it starts to transmit a signal until the end of the first part of the slot. This protocol allows all nodes in the network to know, during each slot, which is the node with the highest priority that has a packet ready for transmission, and thus letting it transmit it successfully.

From the above discussion it is clear that a fully-connected network that applies the protocol suggested in [3] belongs to the class of systems analyzed in this paper.

Example

Let us consider the packet-radio network depicted in Fig. 2. Assume that in this network the three nodes can hear each other and they apply the protocol suggested in [3]. We shall assume that node i has priority i, $1 \leqslant i \leqslant 3$. Packets arrive to the nodes from their corresponding sources. Nodes 2 and 3 transmit their packets to node 1 and node 1 finally transmits all the packets to the outside of the system. Therefore, we have here

$$\theta_1(2) = \theta_1(3) = \theta_3(2) = \theta_2(3) = \theta_3(0) = \theta_2(0) = 0, \qquad \theta_1(0) = \theta_2(1) = \theta_3(1) = 1.$$

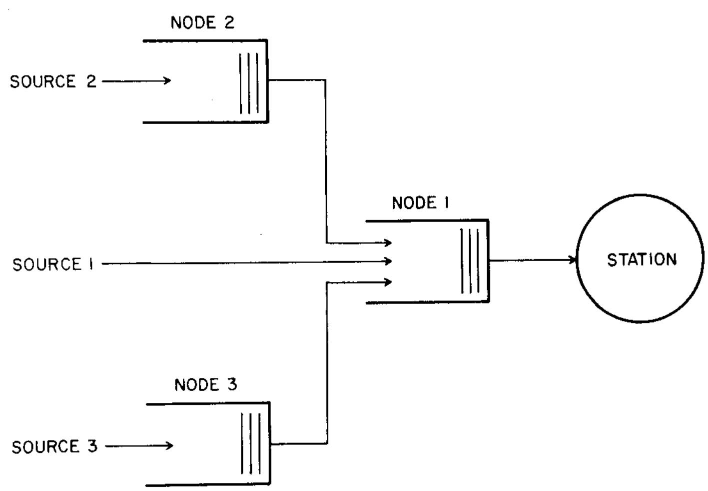

Fig. 2. A three-node packet-radio network.

From (7) we obtain in this case

$$G(z_3, z_2, z_1) = = F(z_3, z_2, z_1)\frac{(z_2^{-1}z_1 - z_1^{-1})G(z_3, z_2, 0) + (z_3^{-1} - z_2^{-1})z_1 G(z_3, 0, 0) + (1 - z_3^{-1}z_1)G(0, 0, 0)}{1 - z_1^{-1}F(z_3, z_2, z_1)}. \tag{25}$$

We can now apply the solution method presented in Section 3. To facilitate the presentation we shall assume that packets do not arrive at node 1 from its corresponding source and that the arrival processes into nodes 2 and 3 are independent Bernoulli processes, i.e.,

$$F(z_3, z_2, z_1) = (z_3 r_3 + \bar{r}_3)(z_2 r_2 + \bar{r}_2), \tag{26}$$

where $\bar{r}_i = 1 - r_i$, $i = 2, 3$, and r_i is the average arrival rate (in units of packets/slot) at node i. Now, from (8a), (13), (16), (18) and (19) we obtain in this case that

$$\hat{z}_1(z_2, z_3) = (z_3 r_3 + \bar{r}_3)(z_2 r_2 + \bar{r}_2), \tag{27}$$

$$\hat{z}_2(z_3) = \left(1 - 2\alpha r_2 \bar{r}_2 - \sqrt{1 - 4\alpha r_2 \bar{r}_2}\right)/(2\alpha r_2^2) \quad \text{where} \quad \alpha = (z_3 r_3 + \bar{r}_3)^2 \tag{28}$$

and

$$G(z_3, 0, 0) = -\frac{1 - z_3^{-1}\hat{z}_1(\hat{z}_2(z_3), z_3)}{[z_3^{-1} - \hat{z}_2(z_3)]\hat{z}_1(\hat{z}_2(z_3), z_3)}G(0, 0, 0), \tag{29}$$

$$G(z_3, z_2, 0) = -\frac{[1 - z_3^{-1}\hat{z}_1(z_2, z_3)]G(0, 0, 0) + (z_3^{-1} - z_2^{-1})\hat{z}_1(z_2, z_3)G(z_3, 0, 0)}{z_2^{-1}\hat{z}_1(z_2, z_3) - \hat{z}_1^{-1}(z_2, z_3)} \tag{30}$$

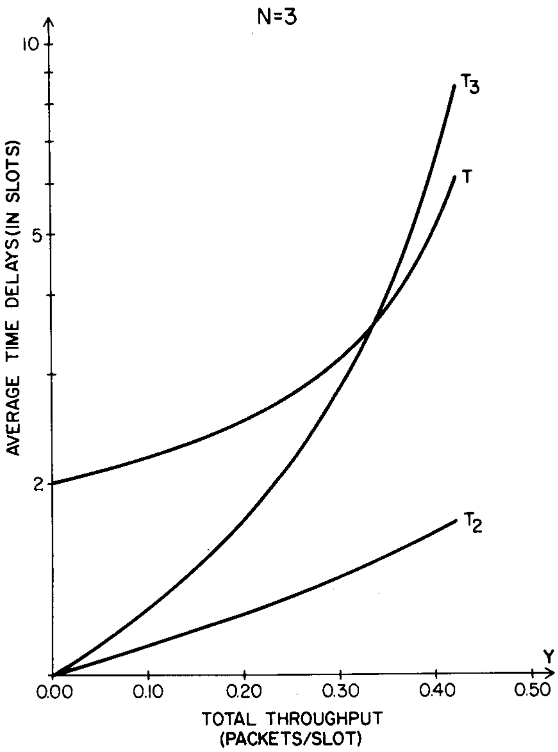

Fig. 3. Time delays vs. γ for the network of Fig. 2, with $r_1 = 0$, $r_2 = r_3 = r$, $\gamma = 2r$.

where

$$G(0, 0, 0) = 1 - 2(r_2 + r_3) \tag{31}$$

and the condition for steady-state is $r_2 + r_3 < \frac{1}{2}$. From (25)–(31) we obtain, after tedious algebra,

$$\bar{L}_1 = r_2 + r_3, \qquad \bar{L}_2 = r_2\left(1 + \frac{r_2 + r_3}{1 - 2r_2}\right), \tag{32a,b}$$

$$\bar{L}_3 = r_3 + \frac{2r_2r_3 + A_2(r_2 + r_s) - (\bar{L}_1 + A_2 - 1)(1 - r_2 - r_3)}{1 - 2r_2 - 2r_3} - 1 \quad \text{where} \quad A_2 = \bar{L}_2 - r_2 \tag{32c,d}$$

and

$$T_1 = 1, \qquad T_i = \bar{L}_i / r_i, \quad 2 \leqslant i \leqslant 3. \tag{33a,b}$$

From (32b) we notice that though node 3 has lower priority than node 2, it affects the delay at node 2 since both nodes 2 and 3 route their packets through node 1. In Fig. 3, T_2 and T_3, the average delays (in units of slots) at nodes 2 and 3 respectively, and the total average delay in the network, T, are plotted vs. the total arrival rate (in units of packets/slot) into the network, γ, when $r_2 = r_3 = r$ (clearly $\gamma = 2r$).

Tandem packet-radio networks

A tandem packet radio network with N nodes is depicted in Fig. 4. In this network all nodes share a common radio channel and are equipped with radio transmitting and receiving devices. We assume that every node can either transmit or receive but not simultaneously. Instantaneous feedback to the transmitter is assumed meaning that a node knows at the end of the slot if the transmitted packet has been successfully received. All nodes are assumed to have full access capability to the common channel. This means that each node always transmits a packet when its buffer is nonempty, while when it is empty, it doese not transmit and is able to receive packets transmitted by other nodes. We also assume that the network topology is such that when node i $(2 \leqslant i \leqslant N - 1)$ transmits only nodes $i + 1$ and $i - 1$ can hear the transmission. When nodes 1 or N transmit, then only nodes 2 and $N - 1$ can hear the transmission respectively. We finally assume that packets leave the network only when they are transmitted by node 1.

Tandem packet-radio networks with arbitrary number of nodes do not belong to the class of systems considered in this paper since in these networks nodes i and $i + 3$ may succeed in their transmissions simultaneously. However, a three-node tandem packet radio network does belong to the class of systems considered in this paper. This is seen by noticing that, in such a network, it is clear that transmissions from node 1 are always successful. Also a packet cannot leave node 2 unless node 1 is empty, because node 1 always transmits when it is nonempty, and in such a case it cannot receive packets from node 2. Finally, a packet may not leave node 3 unless nodes 2 and 1 are both empty. The reason that node 2 should be empty is clear. Node 1 must also be empty, otherwise it transmits, and the transmissions of nodes 1 and 3 interfere, so that the packet transmitted by node 3 cannot be successfully received at node 2. Therefore, this network belongs to the systems considered in this paper. Also, from the network topology we find that $P_3(z) = z_2$; $P_2(z) = z_1$; $P_1(z) = 1$.

Now we can apply the solution method presented in Section 3. In order to give numerical results we

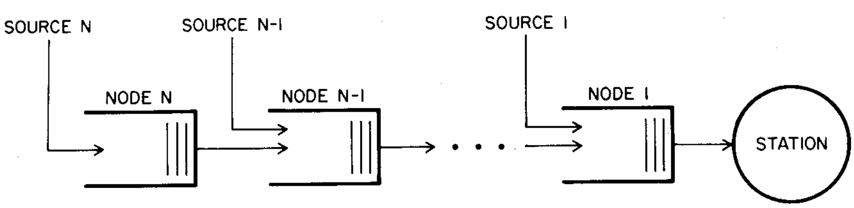

Fig. 4. A tandem packet-radio network with N nodes.

have done it when the arrival processes into nodes 1, 2 and 3 are independent Bernoulli processes with average rates r_1, r_2 and r_3 respectively (in units of packets/slot), i.e.,

$$F(z_3, z_2, z_1) = (z_3 r_3 + \bar{r}_3)(z_2 r_2 + \bar{r}_2)(z_1 r_1 + \bar{r}_1) \quad \text{where } \bar{r}_i = 1 - r_i \text{ for } i = 1, 2, 3. \tag{34}$$

For this case we have obtained explicit expressions for the average number of packets at each node and then by using Little's law we have found T_1, T_2, T_3—the average time delays at nodes 1, 2 and 3 respectively (in units of slots)—and also T—the average time in the system. The results are

$$T_1 = \frac{1}{r_1 + r_2 + r_3}(r_1 + A_1), \tag{35a}$$

$$T_2 = \frac{1}{r_2 + r_3}(r_2 + A_2), \tag{35b}$$

$$T_3 = \frac{1}{r_3}\left(r_3 + \frac{(r_2 + r_3)A_2 + 2r_2 r_3 - (A_1 + A_2)(1 - r_1 - r_2 - r_3) + r_2 + 2r_3 + r_1 r_2 + r_1 r_3}{1 - r_1 - 2r_2 - 3r_3}\right), \tag{35c}$$

$$T = T_1 + \frac{r_2 + r_3}{r_1 + r_2 + r_3} T_2 + \frac{r_3}{r_1 + r_2 + r_3} T_3 \tag{35d}$$

where

$$A_1 = (r_2 + r_3)/(1 - r_1), \tag{35e}$$

$$A_2 = \{r_2(r_2 + r_3) + (1 - r_1)(r_3 + r_1 r_2)\}/\{(1 - r_1)(1 - r_1 - 2r_2)\} \tag{35f}$$

and the condition for steady state is that

$$1 - r_1 - 2r_2 - 3r_3 > 0. \tag{36}$$

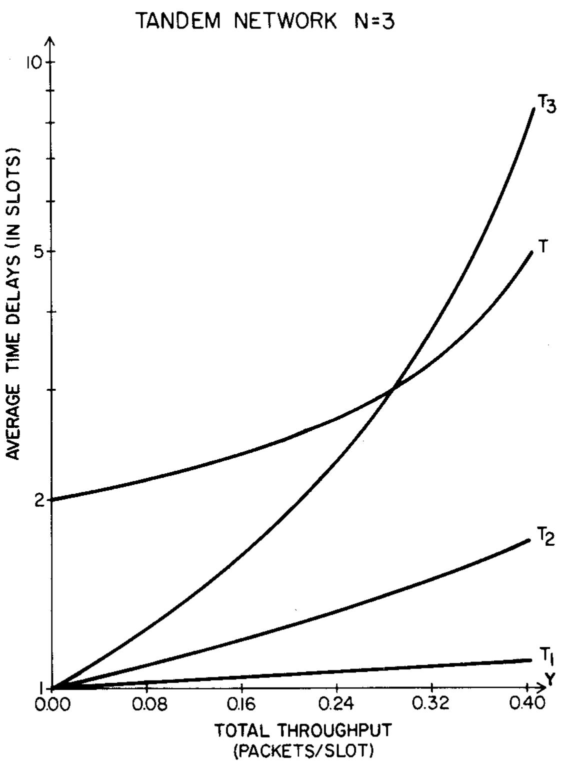

Fig. 5. A three-node tandem packet-radio network: delays vs. γ when $r_1 = r_2 = r_3 = r$, $\gamma = 3r$.

In Fig. 5 the average time delays T_1, T_2, T_3 and T are plotted versus γ—the total throughput when $r_1 = r_2 = r_3 = r$ ($\gamma = 3r$).

Appendix A

Consider the evolution equations (3),

$$L_i(t+1) = L_i(t) + A_i(t) + \sum_{m=1}^{N} D_m^i(t) - U_i(L_i(t)) \prod_{m=1}^{i-1} \left[1 - U_m(L_m(t))\right]. \tag{A.1}$$

Let $G_t(z) = \mathrm{E}[\prod_{i=1}^N z_i^{L_i}(t)]$. Then from (A.1) we have

$$G_{t+1}(z) = \mathrm{E}\left[\prod_{i=1}^{N} z_i^{L_i(t+1)}\right] = \mathrm{E}\left[\prod_{i=1}^{N} z_i^{L_i(t)+A_i(t)+\sum_{m=1}^N D_m^i(t) - U_i(L_i(t))\prod_{m=1}^{i-1}[1-U_m(L_m(t))]}\right]$$

$$= F(z)\mathrm{E}\left[\prod_{i=1}^{N} z_i^{L_i(t)+\sum_{m=1}^N D_m^i(t) - U_i(L_i(t))\prod_{m=1}^{i-1}[1-U_m(L_m(t))]}\right], \tag{A.2}$$

where in (A.2) we used (2) and the fact that the vector arrival processes are independent of the state of the system. Now for $0 \leqslant j \leqslant N$ let the event that $L_i(t) = 0$ for $1 \leqslant i \leqslant j$ and $L_{j+1}(t) > 0$ be denoted by $\mathrm{EVENT}_j(t)$. Then from (A.2) we have

$$G_{t+1}(z) = F(z) \sum_{j=0}^{N} \mathrm{Prob}\left(\mathrm{EVENT}_j(t)\right) \mathrm{E}\left[\prod_{i=1}^{N} z_i^{L_i(t)+\sum_{m=1}^N D_m^i(t) - U_i(L_i(t))\prod_{m=1}^{i-1}[1-U_m(L_m(t))]} / \mathrm{EVENT}_j(t)\right]. \tag{A.3}$$

Now using the definitions of the random variables $U_i(L_i(t))$ and $D_m^i(t)$ for $1 \leqslant i, m \leqslant N$ we have

$$\begin{aligned} G_{t+1}(z) = F(z)\Big\{ &\left[G_t(z) - G_t(z)|_{z_1=0}\right] z_1^{-1}\left[\theta_1(1)z_1 + \theta_1(2)z_2 + \cdots + \theta_1(N)z_N + \theta_1(0)\right] \\ &+ \left[G_t(z)|_{z_1=0} - G_t(z)|_{z_2=z_1=0}\right] z_2^{-1}\left[\theta_2(1)z_1 + \theta_2(2)z_1 + \cdots + \theta_2(N)z_N + \theta_2(0)\right] \\ &+ \cdots + \left[G_t(z)|_{z_{N-1}=z_{N-2}=\cdots=z_1=0} - G_t(z)|_{z_N=z_{N-1}=\cdots=z_1=0}\right] \\ &\times z_N^{-1}\left[\theta_N(1)z_1 + \theta_N(2)z_2 + \cdots + \theta_N(N)z_N + \theta_N(0)\right] + G_t(z)|_{z_N=z_{N-1}=\cdots=z_1=0}\Big\}. \end{aligned} \tag{A.4}$$

Letting $t \to \infty$ in (A.4) we obtain (7).

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