Splitting Algorithms in Noisy Channels with Memory

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Abstract —Multiaccess networks are considered in which the shared channel is noisy. We assume a slotted-time collision-type channel, Poisson infinite-user model, and binary feedback. Due to the noise in the shared channel, the received signal may be detected as a collision even though no message or a single message is transmitted. This kind of imperfect feedback is referred to as *error*. A common assumption in all previous studies of multiaccess algorithms in channels with errors is that the channel is *memoryless*. We consider the problem of splitting algorithms when the channel is *with memory*. We introduce a two-state first-order Markovian model for the channel and analyze the operation of the tree collision-resolution algorithm in this channel. We obtain a stability result, i.e., the necessary conditions on the channel parameters for stability of the algorithm. Assuming that the stability conditions hold, we calculate the throughput of the algorithm. Extensions to more general channel models are discussed.

I. Introduction

IN A MULTIACCESS network many independent geographically distributed users share a common communication channel. During recent years, splitting algorithms have received considerable attention as a means of coordinating the access of the users to the shared channel. The common model for a multiaccess network considered in studies of splitting algorithms consists of the following assumptions [13]: 1) an infinite number of independent users transmitting messages of equal length over a slotted-time collision-type channel; 2) the number of messages generated by all users collectively in each slot is Poisson distributed with mean λ; and 3) binary feedback, i.e., at the end of each slot each user obtains a feedback which indicates whether a collision has occurred in that slot (at least two messages were transmitted), or whether there was no collision in the slot (no message or a single message was transmitted).

In most studies it is assumed that the common channel used by the users for transmitting their messages (the forward channel) is free of interferences that can cause incorrect feedback. In practical radio channels, however, the received signal may be detected as a collision even though either no message or only a single message is transmitted. Possible reasons for this are atmospheric noise, multipath fading, enemy jamming, etc. This kind of imperfect feedback is referred to as *error*.

Splitting algorithms in channels with errors were considered in [1]–[5]. The algorithms in [1]–[3] are based on the tree collision-resolution algorithm (CRA) [6], [7]. The algorithm in [4] is based on the 0.487 algorithm [8], [9], and the algorithm in [5] is a stack algorithm [10].

A common assumption in all previous studies of multiaccess algorithms in channels with errors is that the channel is *memoryless*, i.e., given the number of messages transmitted in each slot, error events in different slots are independent (each occurs with the same probability). This is, of course, a simplifying assumption since the phenomena mentioned above as possible causes of errors can be *with memory*. The effects of memory on splitting algorithms, as well as on multiaccess systems in general, are not known. In particular, it is not clear what the effect of memory on the stability of the algorithms is, and how the memory affects the throughput of the algorithms within the stable region.

In this paper, we examine the operation of the tree CRA in channels with errors when the channel is with memory. In Section II we introduce our model for the channel and give a concise description of the tree CRA. In Sections III–V we analyze the operation of the tree CRA in this channel. Finally, in Section VI we make some remarks about possible extensions of the analysis to more general channel models.

II. The Model

We adapt the common assumptions 1)–3) (presented in Section I) to our model. In this section we introduce our model for the channel and give a concise description of the tree CRA [1].

A. The Channel Model

An *error* is defined as the event in which when either no message or only a single message is transmitted, the received signal is detected as a collision, and therefore all users obtain a "collision" feedback. Note that if an error occurs when a single message is transmitted, the transmitting user knows that his message was not received successfully and therefore must retransmit it at some later time.

Our channel model, which is a version of Gilbert channel [12], is as follows (extensions are discussed in Section VI). In each slot the channel can be in one of two states–

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b (for "bad") and g (for "good"). When the channel is in state b, if either no message or only a single message is transmitted, then an error occurs with probability ρ_0 or ρ_1, respectively. When the channel is in state g, no errors occur (with probability 1). Let X_n be the state of the channel at time (slot) n. The sequence $\{X_n, n \geq 0\}$ is assumed to be a homogeneous Markov chain with state space $E = \{b, g\}$ and transition matrix

$$P_x = \begin{bmatrix} r_g & 1-r_g \\ r_b & 1-r_b \end{bmatrix}.$$

The elements of this matrix will be denoted by $p_x(i, j)$, $i, j \in E$. The state diagram of this chain is shown in Fig. 1.

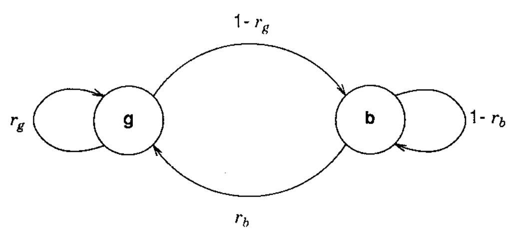

Fig. 1. State diagram of channel.

Note that with the above notation, when the channel is in state z, $z \in E$, the next state is g with probability r_z and b with probability $1-r_z$. This will henceforth facilitate the notation.

When $1-r_g$ and r_b are relatively small, the model simulates a bursty channel. When $r_g = r_b \triangleq r$, the model corresponds to a memoryless channel where the probability of an error is $(1-r)\rho_0$ or $(1-r)\rho_1$, according to whether no message or a single message is transmitted in the slot.

B. The Tree Collision Resolution Algorithm [1]

Consider two time axes. The first, called the arrival axis, shows the Poisson arrival instants. The second, called the transmission axis, is segmented into consecutive intervals called *collision resolution intervals* (CRI's), each consisting of an integral number of slots. At the first slot of the ith CRI, the algorithm enables an interval $E_i = [a_i, b_i)$ of the arrival axis. If no collision occurs in this slot, the ith CRI terminates. If a collision occurs in this slot, say the kth slot, all users involved in the collision split into two subsets (e.g., by each flipping a coin), in such a way that a user belongs to the first subset with probability $q \in (0,1)$. The first subset transmits in slot $k+1$, and if there is no collision in that slot, the second subset transmits in slot $k+2$; if a collision occurs in slot $k+1$, then the first of the two subsets splits again, and the second subset transmits in the next slot after the collision among the users of the first subset has been resolved. A collision is said to be resolved at the moment when all users know that the colliding messages have all been successfully retransmitted. The ith CRI terminates at the moment when the collision among the users that transmitted in its first slot has been resolved. The $(i+1)$st CRI starts in the next slot after the ith CRI has terminated. The interval $E_i = [a_i, b_i)$ is determined as follows: $a_0 = 0$, $b_0 = \Delta$, and for all $i \geq 1$, $a_i = b_{i-1}$, $b_i = \min(b_{i-1} + \Delta, t_c)$, where t_c is the current time and Δ is a parameter (that is chosen so that the algorithm is optimal with respect to some criterion, e.g., throughput). In this way the arrival axis is entirely searched and resolved.

It is well-known that the evolution of each CRI can be represented by a binary tree [1]. Also, the above tree CRA can be described equivalently as a stack algorithm [15]. This description is the same as before except that now the mechanism for resolving collisions is described as follows. Each user independently utilizes a counter. A user that has a message transmits it if and only if the value of his counter is zero. When the user is first enabled the counter is set to zero, and is afterwards updated at the end of each slot according to the feedback obtained, as follows. If the value of the counter is greater then zero, then the user increments it by one for a "collision" feedback and decrements it by one for a "noncollision" feedback. If the value of the counter is zero, then a "noncollision" feedback indicates that the message was successfully transmitted and the user thus leaves the system; if a "collision" feedback is obtained, then with probability q the value of the counter is not changed and with probability $1-q$ it is incremented by one. Now, consider an infinite stack with cells numbered $0, 1, 2, \cdots$. The above counters can be referred to as pointers to this stack. Thus one can refer to the number of messages in each cell at every slot, and the algorithm can be completely described by the content of the stack at each slot.

III. Preliminaries

In the next two sections we analyze the operation of the tree CRA in the channel introduced in Section II. We first consider the special case $\rho_0 = \rho_1 = 1$, since the simplicity of this case enables one to get more insight into the problem. Then we generalize the results for the case $\rho_0 = \rho_1 = \rho$ where $0 < \rho \leq 1$, and thereafter give the modifications needed to extend the results for the case $\rho_0 \neq \rho_1$ where $0 < \rho_0, \rho_1 \leq 1$.

The analysis proceeds in two steps. In the first step, a stability result is obtained. That is, we derive the necessary conditions on the channel parameters r_g, r_b, and ρ_i for stability of the algorithm. In the second step, assuming that the stability conditions hold, we calculate the throughput of the algorithm.

Throughout the analysis, we assume that $0 < r_g, r_b < 1$. The cases where this assumption does not hold are degenerate. Some of these cases are not interesting (for instance, $r_b = 0$ or $r_g = 1$) and others (for instance, $r_b = 1, r_g > 0$) can be handled easily (using the ideas and techniques presented).

We now present some additional notation. Consider the stack that represents the evolution of the algorithm. Let $y_n^{(i)}$ be the number of messages in the ith cell of the stack at slot n, and for all $n \geq 0$ denote $Y_n = (y_n^{(0)}, y_n^{(1)}, y_n^{(2)}, \cdots)$. The process $\{(X_n, Y_n),\ n \geq 0\}$ is a Markov chain satisfying

$$P(X_{n+1}=j,\ Y_{n+1}=y \mid X_n=i,\ Y_n=y') = p_x(i,j) P(Y_{n+1}=y \mid X_n=i,\ Y_n=y') \quad (3.1)$$

for every n, i, j, y, y'. Equation (3.1) reflects the fact that the evolution of the channel states is not influenced by the algorithm.

IV. The Special Case $\rho_0 = \rho_1 = 1$

The case $\rho_0 = \rho_1 = 1$ corresponds to the situation in which whenever the channel is in state b and either no message or a single message is transmitted, an error occurs. The unique feature of this case is that in the last slot of each CRI or sub-CRI the channel state is g, and therefore at the beginning of each CRI or sub-CRI the channel state is g (or b) with the same constant probability, independent of the past.

A. Stability

Consider the binary tree that represents the evolution of some CRI in an error-free channel. It is well-known that the expected number of vertices of this tree is finite. When there are errors in the channel, the binary tree that represents the evolution of the CRI can differ from the error-free tree if and only if an error occurs in a slot that corresponds to a leaf of the error-free tree, in which case a new subtree evolves from that leaf. We shall refer to such a subtree as an *error tree*. It follows that the expected number of vertices in the binary tree that represents the evolution of the CRI is finite if and only if the expected number of vertices of each error tree is finite. Thus, for any $k \geq 1$, consider the kth error tree that evolves during the operation of the algorithm, and denote by T_k the slot that corresponds to its root. Let l_k be the length of the sub-CRI that corresponds to this error tree. From the operation of the algorithm, it follows that given X_{T_k}, the random variable l_k depends only on the channel states during that sub-CRI and is independent of the coin tosses and the channel states in the past (before T_k). Since $X_{T_m} = b$ for all $m \geq 1$, it follows that the random variables $\{l_m, m \geq 1\}$ are identically distributed.

We wish now to obtain the distribution of the channel states in certain slots of the sub-CRI, as follows. The state of the channel in the slot that corresponds to the root of the error tree is b and hence the root has two sons. We shall refer to the subtree that evolves from the son that corresponds to slot $T_k + 1$ as the right subtree, and to the other subtree as the left subtree. Since the slot that corresponds to the root of the right subtree is $T_k + 1$, the state of the channel in this slot is g with probability $p_x(b, g) = r_b$, and b with probability $p_x(b, b) = 1 - r_b$. Observe that if the state in slot $T_k + 1$ is b, then the right subtree is again an error tree.

To determine the distribution of the channel states in the slot that corresponds to the root of the left subtree, we observe that similar arguments *cannot* be employed. The reason is that the number of slots that elapse between T_k and the slot that corresponds to the root of the left subtree is a random variable which is determined by the evolution of the algorithm and the channel states up to that slot. Thus let T_k' be the slot that corresponds to the root of the left subtree. As will be seen shortly, it is enough to consider only the *finite* case, i.e., when $T_k' < \infty$. Thus suppose that $T_k' < \infty$. Now, any sub-CRI must terminate with a "noncollision" feedback, and therefore we have $X_{T_k'-1} = g$. Since $T_k' - 1$ is a stopping time for $\{X_n, n \geq 0\}$, it follows by the strong Markov property of $\{X_n, n \geq 0\}$ that $P(X_{T_k'} = j) = p_x(g, j)$, $j \in E$. Again, note that if the state in slot T_k' is b, then the left subtree is an error tree.

Let ϕ be the probability that the length of the sub-CRI that corresponds to the kth error tree is finite, i.e., $\phi = P(l_k < \infty)$. We have

$$\phi = [r_b + (1 - r_b)\phi][r_g + (1 - r_g)\phi]. \quad (4.1)$$

The explanation for (4.1) is the following. The error tree is finite if and only if the right and left subtree are both finite. The first term is the probability that the right subtree is finite. This occurs if and only if *either* 1) *or* 2) occurs: 1) the state of the channel in the slot that corresponds to the root is g (this occurs with probability r_b); 2) the state of the channel in the slot that corresponds to the root is b (this occurs with probability $1 - r_b$) *and* the subtree that evolves from the root is finite (given that the subtree starts in state b this occurs with probability ϕ since the subtree is again an error tree). The second term is the probability that the left subtree is finite *given* that the right subtree is finite. This occurs if and only if *either* 1) *or* 2) occurs, except that now the probabilities are r_g and $(1 - r_g)\phi$, respectively.

Solving (4.1) for ϕ we obtain $\phi_1 = 1$ and $\phi_2 = r_b r_g / ((1 - r_b)(1 - r_g))$. Since ϕ must satisfy $0 \leq \phi \leq 1$, it follows that a sufficient condition for $\phi = 1$ is $r_b r_g \geq (1 - r_b)(1 - r_g)$. Let $\nu(b)$ and $\nu(g)$ be the invariant probabilities of $\{X_n, n \geq 0\}$; we have $\nu(b) = (1 - r_g)/(1 - r_g + r_b)$ and $\nu(g) = 1 - \nu(b)$. Then the last inequality is equivalent to $\nu(b) \leq 1/2$. Thus we have proved the following.

Proposition 1: If $\nu(b) \leq 1/2$, then $P(l_k < \infty) = 1$.

We wish now to calculate $L^b \triangleq E(l_k)$, i.e., the expected length of the sub-CRI that corresponds to the kth error tree. To that end, suppose first that $P(l_k < \infty) = 1$. Then we have

$$L^b = 1 + r_b[1 + r_g + (1 - r_g)L^b] + (1 - r_b)[L^b + r_g + (1 - r_g)L^b]. \quad (4.2)$$

The explanation for (4.2) is the following. The 1 is the first slot of the sub-CRI. The second term corresponds to the case in which the state of the channel in the second slot of the sub-CRI is g. This occurs with probability r_b, and then 1) the (expected) length of the sub-CRI that corresponds to the right subtree is 1; 2) the expected length of the sub-CRI that corresponds to the left subtree is 1 if the state of the channel in its first slot is g (this occurs with probability r_g), and is L^b if the state of the channel is b (this occurs with probability $(1-r_g)$). The third term corresponds to the case in which the state of the channel in the second slot of the sub-CRI is b. This occurs with probability $1-r_b$ and then 1) and 2) above apply except that now the 1 in 1) should be replaced by L^b.

Now, if $P(l_k<\infty)<1$, then $L^b=\infty$ and (4.2) is still satisfied. Thus (4.2) is always true, regardless of the value of $P(l_k<\infty)$.

Equation (4.2) is of the form $x=x(2-r_g-r_b)+(r_g+r_b+1)$. The solutions of this equation are $x_0=-\infty$, $x_1=\infty$, $x_2=(r_g+r_b+1)/(r_g+r_b-1)=1+1/[(1-r_g+r_b)(1/2-\nu(b))]$, where the last solution is valid if and only if $\nu(b)\neq 1/2$. The algorithmic parameter L^b is a solution of the equation that satisfies $L^b\geq 1$. Thus $L^b\neq x_0$ and it remains to determine to which of the the two solutions, x_1 or x_2, the algorithmic parameter L^b corresponds. We have the following.

Theorem 1: If $\nu(b)<1/2$, then

$$L^b=1+\frac{1/(1-r_g+r_b)}{1/2-\nu(b)}<\infty.$$

If $\nu(b)\geq 1/2$, then $L^b=\infty$.

The proof of Theorem 1 is given in the Appendix.

Let τ_i $(i\geq 0)$ be the last slot of the ith CRI. As noted before, the expected number of vertices of the binary tree that represents the evolution of any CRI is finite if and only if the expected number of vertices of each error tree is finite. Recalling that $L^b=E(l_k)$ for all $k\geq 1$, we have the following.

Corollary 1: $E(\tau_i)<\infty$ for all $i\geq 0$ if and only if $\nu(b)<1/2$.

The algorithm is said to be stable if it has a positive throughput (see the next subsection). A necessary condition for the stability of the algorithm is $E(\tau_i)<\infty$ for all $i\geq 0$, since otherwise the expected length of some CRI is infinite and hence the throughput is zero. Thus we have the following.

Corollary 2: If $\nu(b)\geq 1/2$, then the algorithm is unstable for all $\lambda>0$.

As will be evident later, Corollary 2 holds also if $\rho_1\in[0,1]$ (and $\rho_0=1$). Thus this is a generalization of the well-known result in the memoryless case: the algorithm is unstable if the probability that an error occurs in an idle slot is equal to or greater than 1/2 (recall that the memoryless model is a special case of our Markovian model).

Remark: Corollary 1 implies that if $\nu(b)<1/2$, then $P(\tau_i<\infty)=1$ for all $i\geq 0$. It can be further shown that $P(\tau_i<\infty)=1$ for all $i\geq 0$ if and only if $\nu(b)\leq 1/2$.

B. Throughput

By Corollary 1, the expected length of each CRI is finite if and only if $\nu(b)<1/2$. We now assume that this condition holds and proceed to calculate the throughput of the algorithm. For any $i\geq 0$ consider the ith CRI. Let a_i be the number of messages that were transmitted in the first slot (T_i) of this CRI, and let l_i be its length. It is clear that given a_i and X_{T_i}, the random variable l_i depends only on the results of the coin tosses performed in the algorithm and the channel states during that CRI, and is independent of the coin tosses and the channel states in the past. For $z\in E$, denote $L_n^z=E[l_i|a_i=n,\ X_{T_i}=z]$; i.e., L_n^z is the expected length of the ith CRI given that n messages are transmitted in its first slot and the state of the channel in this slot is z. We have

$$L_0^g=L_1^g=1\qquad L_0^b=L_1^b=L^b=\frac{r_g+r_b+1}{r_g+r_b-1}\tag{4.3}$$

$$L_n^z=1+\sum_{j=0}^{n}Q_n(j)\Big\{r_z\big[L_j^g+r_gL_{n-j}^g+(1-r_g)L_{n-j}^b\big]+(1-r_z)\big[L_j^b+r_gL_{n-j}^g+(1-r_g)L_{n-j}^b\big]\Big\},\qquad n\geq 2,\ z\in E\tag{4.4}$$

where

$$Q_n(j)=\binom{n}{j}q^j(1-q)^{n-j}.$$

The explanation for (4.4) is the following. The 1 is the first slot of the CRI. At the end of this slot, the n colliding messages split into two subsets in such a way that with probability $Q_n(j)$ the first subset contains j messages and the second subset contains $n-j$ messages. In the second slot of the CRI, the first subset is transmitted. The state of the channel in this slot is g with probability r_z and b with probability $1-r_z$. If it is g, it takes L_j^g slots on average to resolve the collision among the j messages. It then takes L_{n-j}^g or L_{n-j}^b slots on average to resolve the collision among the $n-j$ messages of the second subset, according to the channel state in the first slot (denote this slot by T) of the resolution of this collision. As noted before, any sub-CRI must terminate with a "noncollision" feedback and hence $X_{T-1}=g$. Since $T-1$ is a stopping time for $\{(X_n,Y_n),\ n\geq 0\}$, it follows by the strong Markov property of $\{(X_n,Y_n),\ n\geq 0\}$ and (3.1) that $P(X_T=g)=r_g$ and $P(X_T=b)=1-r_g$. The last term is explained similarly.

By rearranging terms in (4.4) we obtain a somewhat simpler expression:

$$L_n^z=1+\sum_{j=0}^{n}Q_n(j)\big[r_zL_j^g+(1-r_z)L_j^b+r_gL_{n-j}^g+(1-r_g)L_{n-j}^b\big],\qquad n\geq 2,\ z\in E.\tag{4.5}$$

Solving (4.5) for (L_n^g, L_n^b), we obtain

$$L_n^g=\frac{C_1B_2-C_2B_1}{A_1B_2-A_2B_1}\qquad L_n^b=\frac{A_1C_2-A_2C_1}{A_1B_2-A_2B_1}\tag{4.6}$$

where

$$A_1 = 1 - [Q_n(0) + Q_n(n)] r_g$$
$$B_1 = -[Q_n(0) + Q_n(n)](1 - r_g)$$
$$C_1 = 1 + [Q_n(0) + Q_n(n)][r_g + (1 - r_g)L_0^b] + \sum_{j=1}^{n-1} Q_n(j)[r_g(L_j^g + L_{n-j}^g) + (1 - r_g)(L_j^b + L_{n-j}^b)]$$
$$A_2 = -[Q_n(0)r_g + Q_n(n)r_b]$$
$$B_2 = 1 - Q_n(0)(1 - r_g) - Q_n(n)(1 - r_b)$$
$$C_2 = 1 + Q_n(0)[r_b + (1 - r_b)L_0^b] + Q_n(n)[r_g + (1 - r_g)L_0^b] + \sum_{j=1}^{n-1} Q_n(j)[r_b L_j^g + (1 - r_b)L_j^b + r_g L_{n-j}^g + (1 - r_g)L_{n-j}^b].$$

Equation (4.6) is a recurrence equation for which the initial conditions are given by (4.3).

Now, let $L(z) = E[l_i | X_{T_i} = z]$. Then the Poisson arrivals imply that

$$L(z) = \sum_{n=0}^{\infty} \frac{e^{-\lambda\Delta}(\lambda\Delta)^n}{n!} L_n^z, \qquad z \in E. \tag{4.7}$$

Let $l(k)$ and $m(k)$, $k \geq 1$, be the length of the k first CRI's and the number of successfully transmitted messages during them, respectively. The throughput of the algorithm is defined as $T = \lim_{k \to \infty}\{E[m(k)]/E[l(k)]\}$. Let m_i, $i \geq 0$, be the number of successfully transmitted messages during the ith CRI. From the operation of the algorithm it follows that $E(m_i) = \lambda\Delta$ for all $i \geq 0$. Let T_i ($i \geq 0$) be the first slot of the ith CRI ($T_0 = 0$). That is, $T_i = \tau_{i-1} + 1$ for all $i \geq 1$ (recall that τ_i is the last slot of the ith CRI). Each CRI must terminate with a "noncollision" feedback, and therefore $X_{\tau_i} = g$ for all $i \geq 0$. Since for all $i \geq 0$, τ_i is a stopping time for $\{(X_n, Y_n),\ n \geq 0\}$, it follows by the strong Markov property of $\{(X_n, Y_n),\ n \geq 0\}$ and (3.1) that $P(X_{T_i} = j) = p_x(g, j)$, $j \in E$, for all $i \geq 1$. In other words, $\{X_{T_i},\ i \geq 1\}$ is an independent identically distributed (i.i.d) process. Hence, for all $i \geq 1$,

$$E(l_i) = E[E(l_i | X_{T_i})] = E[L(X_{T_i})] = r_g L(g) + (1 - r_g)L(b).$$

Thus the throughput is

$$T = \lim_{k \to \infty} \frac{E\left[\sum_{i=1}^{k} m_i\right]}{E\left[\sum_{i=1}^{k} l_i\right]} = \lim_{k \to \infty} \frac{k(\lambda\Delta)}{k \cdot [r_g L(g) + (1 - r_g)L(b)]} = \frac{\lambda\Delta}{r_g L(g) + (1 - r_g)L(b)}.$$

Numerical results are given in Fig. 2, where the throughput is plotted versus r_b for various values of $\nu(b)$ (for

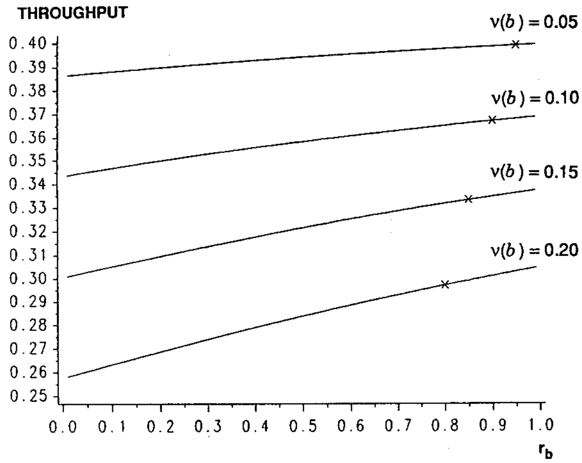

Fig. 2. Maximum throughput versus r_b for various values of $\nu(b)$. Marked points correspond to memoryless case (i.e., $r_b = r_g = 1 - \nu(b)$).

$g = 1/2$). We observe that for a constant value of $\nu(b)$, the throughput increases as r_b increases, and vice versa. Thus, for example, for values of r_b that are higher than the corresponding value in the memoryless case (i.e., $1 - \nu(b)$), higher throughput than that in the memoryless case is achieved. In other words, the shorter the average time the channel spends in state b once it enters this state, the higher the throughput achieved. An intuitive explanation for this phenomenon is the following. Consider the binary tree that corresponds to some CRI in an error-free channel. The channel can enter state b in a slot that corresponds either to an internal vertex or to a leaf, but only in the latter case will there be a degradation in the throughput. If the time the channel spends in state b is long enough, then even if this state is entered in a slot that corresponds to an internal vertex, the burst will reach a slot that corresponds to some leaf. Moreover, if the channel is in state b in a slot that corresponds to a leaf, then a consecutive b state occurs also in a slot that corresponds to a leaf, and each "damaged" leaf reproduces at least two more leaves—a sort of an avalanche phenomenon. Note that, for a constant $\nu(b)$, a shorter average time spent in state b once it is entered implies also a shorter average time spent in state g once it is entered.

V. The General Case

In the general case, the state of the channel in the last slot of each CRI (or sub-CRI) is not necessarily g as it is in the special case $\rho_0 = \rho_1 = 1$, and therefore the analysis is somewhat more complicated.

A. The Case $\rho_0 = \rho_1 = \rho$ where $0 < \rho \leq 1$

We first note that the channel model can be described equivalently as follows. When the channel is in state b, an error occurs if and only if $V_n = 1$, where $\{V_n,\ n \geq 0\}$ is a Bernoulli process with parameter ρ which is independent of $\{X_n,\ n \geq 0\}$. We shall hereafter be using this description.

For any $k \geq 1$ consider the kth error tree that evolves during the operation of the algorithm. As in the previous section, we need the distribution of the channel states in the slots that correspond to the roots of the right and the left subtrees of the error tree. For the right subtree we have, as in the previous section, $P(X_{T_k+1}=j)=p_x(b,j)$, $j \in E$, since the state of the channel in the slot that corresponds to the root of the error tree is b. The distribution of the channel states in the slot that corresponds to the root of the left subtree is *not* as in the previous section, since in this case the state of the channel in the last slot of each sub-CRI is not necessarily g. Thus let us first obtain an expression for the probability that the channel state in the last slot of the sub-CRI that corresponds to the error tree is z, $z \in E$. Denote this probability by $w^b(z)$. We have

$$w^b(g)=\rho\{[r_b+(1-r_b)w^b(g)][r_g+(1-r_g)w^b(g)] + (1-r_b)w^b(b)[r_b+(1-r_b)w^b(g)]\}. \quad (5.1)$$

Equation (5.1) is explained in a way similar to (4.1), where we note that the error tree terminates in state g if and only if an error occurs in the slot that corresponds to its root, its right subtree terminates either in state g or in state b, and its left subtree terminates in state g. Here the strong Markov property of $\{(X_n, V_n),\ n \geq 0\}$ is used.

Clearly, $P(l_k<\infty)=w^b(g)+w^b(b)$. Substituting $w^b(b)=\phi-w^b(g)$ in (5.1) yields the following equation:

$$a[w^b(g)]^2+bw^b(g)+c(\phi)=0 \quad (5.2a)$$

where

$$a=\rho(1-r_b)(r_b-r_g),$$
$$b=\rho[(1-r_b)(1-2r_b+r_g)+r_b(1-r_g)]-1,$$
$$c(\phi)=\rho r_b[(1-r_b)\phi+r_g]. \quad (5.2b)$$

We also have

$$\phi=(1-\rho)+\rho\{r_br_g+r_b(1-r_g)\phi + (1-r_b)[w^b(g)(r_g+(1-r_g)\phi) + (\phi-w^b(g))(r_b+(1-r_b)\phi)]\}. \quad (5.3)$$

Equation (5.3) is explained in a way similar to (4.1) and (5.1). Rearranging terms in (5.3) we obtain

$$A\phi^2+B(w^b(g))\phi+C(w^b(g))=0 \quad (5.4a)$$

where

$$A=\rho(1-r_b)^2, \qquad B(w^b(g))=\rho[r_b(2-r_b-r_g) + w^b(g)(1-r_b)(r_b-r_g)]-1,$$
$$C(w^b(g))=1+\rho[r_br_g-1+w^b(g)(1-r_b)(r_g-r_b)]. \quad (5.4b)$$

We consider first the case $r_g \neq r_b$. Solving (5.4) for ϕ we obtain $\phi_1=1$ and $\phi_2=C(w^b(g))/A$. Substituting $\phi=C(w^b(g))/A$ in (5.2) yields

$$\alpha[w^b(g)]^2+\beta w^b(g)+\gamma=0 \quad (5.5)$$

where $\alpha=a(1-r_b)$, $\beta=b(1-r_b)-ar_b$, $\gamma=r_b[1-\rho(1-r_g)]$, and a,b are given by (5.2b). Thus the system of equations

$$ax^2+bx+c(y)=0$$
$$Ay^2+B(x)y+C(x)=0,$$

where $a,b,c(\cdot),A,B(\cdot),C(\cdot)$ are given by (5.2b) and (5.4b), has the following four solutions:

$$x_1=-\left(b+\sqrt{b^2-4ac(1)}\right)/(2a), \qquad y_1=1$$
$$x_2=-\left(b-\sqrt{b^2-4ac(1)}\right)/(2a), \qquad y_2=1$$
$$x_3=-\left(\beta+\sqrt{\beta^2-4\alpha\gamma}\right)/(2\alpha), \qquad y_3=C(x_3)/A$$
$$x_4=-\left(\beta-\sqrt{\beta^2-4\alpha\gamma}\right)/(2\alpha), \qquad y_4=C(x_4)/A. \quad (5.6)$$

Lemma 1: a) For $0 \leq y \leq 1$ the equation $ax^2+bx+c(y)=0$ has two real roots; for $r_g>r_b (r_g<r_b)$ the greatest (smallest) root is in $(0,1]$, and the other root is not in $[0,1]$.
b) The numbers x_4 and y_4 given by (5.6) are not both in the interval $[0,1]$.

The proof of Lemma 1 is given in the Appendix.

Since ϕ and $w^b(g)$ must both be in the interval $[0,1]$, it follows by Lemma 1 that either $\phi=1$ and $w^b(g)=x_1$ or $\phi=C(x_3)/A$ and $w^b(g)=x_3$. Hence a sufficient condition for $P(l_k<\infty)=1$ and $w^b(g)=x_1$ is $C(x_3)-A \geq 0$. (If $C(x_3)-A=0$, then $\phi=1$; hence by (5.2) and Lemma 1-a), we have $w^b(g)=x_1$.) Substituting for x_3 the expression given by (5.6) yields $C(x_3)-A=F+G$ where $F \triangleq 1+\beta/(2(1-r_b))-\rho[1-r_br_g+(1-r_b)^2]$ and $G \triangleq \sqrt{\beta^2-4\alpha\gamma}/(2(1-r_b))$. By straightforward calculations, we obtain $(F+G)(F-G)=D[1/2-\rho\nu(b)]$ where $D=-2\rho(1-r_b)(1-r_g+r_b)<0$. Since $F+G \geq F-G$ it follows that if $\rho\nu(b) \leq 1/2$ then $F+G \geq 0$, i.e., $C(x_3)-A \geq 0$.

Consider now the case $r_g=r_b=r$. In this case $B(\cdot)$ and $C(\cdot)$ in (5.4) are constants and we have $C-A=1-2\rho(1-r)=1-2\rho\nu(b)$. Therefore, if $\rho\nu(b) \leq 1/2$ then $C-A \geq 0$ and hence $\phi=1$. Substituting $\phi=1$ in (5.2) we obtain $w^b(g)=\rho r/(1-\rho(1-r))$.

Thus we have proved the following proposition.

Proposition 2: If $\rho\nu(b) \leq 1/2$, then $P(l_k<\infty)=1$ and

$$w^b(g)=\begin{cases} -\dfrac{1}{2a}\left(b+\sqrt{b^2-4ac(1)}\right), & r_g \neq r_b \\ \dfrac{\rho r}{1-\rho(1-r)}, & r_g=r_b=r \end{cases}$$

where $a,b,c(\cdot)$ are given by (5.2b).

As in the previous section, to obtain an expression for L^b we first suppose that $P(l_k<\infty)=1$. Then we have

$$\begin{aligned} L^b=1+\rho\Big\{ & r_b\big[1+r_g+(1-r_g)L^b\big] \\ & +(1-r_b)\big[L^b+w^b(g)\big[r_g+(1-r_g)L^b\big] \\ & +(1-w^b(g))\big[r_b+(1-r_b)L^b\big]\big]\Big\}. \end{aligned} \tag{5.7}$$

Equation (5.7) is explained in a way similar to (4.2) and (5.1). As in the previous section, (5.7) holds always, regardless of the value of $P(l_k<\infty)$. Solving (5.7) we obtain that either $L^b=\infty$ or $L^b=1+2\rho/(C(w^b(g))-A)\triangleq U$ where $C(\cdot)$ and A are given by (5.4b) (the second solution is valid if and only if $C(w^b(g))-A\neq 0$). Now, if $P(l_k<\infty)<1$, then clearly $L^b=\infty$. Therefore, it suffices to consider the value of U only when $P(l_k<\infty)=1$. We have the following lemma.

Lemma 2: If $P(l_k<\infty)=1$, then $C(w^b(g))-A=H[1/2-\rho\nu(b)]$, where $H=\{-2\rho(1-r_b)(1-r_g+r_b)\}/\{1+b/2-\rho[1-r_br_g+(1-r_b)^2]-\sqrt{b^2-4ac(1)}/2\}>0$ for $r_g\neq r_b$ and $H=2$ for $r_g=r_b$.

The proof of Lemma 2 is given in the Appendix.

Employing the same approach as in the previous section, we obtain Theorem 2.

Theorem 2: If $\rho\nu(b)<1/2$, then

$$L^b=1+\frac{2\rho}{H[1/2-\rho\nu(b)]}<\infty$$

where $H>0$ is given in Lemma 2. If $\rho\nu(b)\geq 1/2$, then $L^b=\infty$.

The proof of Theorem 2 (which is omitted) is similar to the proof of Theorem 1, where we note that 1) In the cases $\rho\nu(b)<1/2$ and $\rho\nu(b)=1/2$ we have, by Proposition 2, that $P(l_k<\infty)=1$; hence by Lemma 2 it follows that in the former case $U=1+2\rho/[H(1/2-\rho\nu(b))]$ and in the latter case $L^b=\infty$. 2) In the case $\rho\nu(b)>1/2$, if $P(l_k<\infty)=1$, then it follows from Lemma 2 that $U=1+2\rho/[H(1/2-\rho\nu(b))]<1$, and if $P(l_k<\infty)<1$ then clearly $L^b=\infty$.

By similar arguments as in the previous section we have two corollaries.

Corollary 3: $E(\tau_i)<\infty$ for all $i\geq 0$ if and only if $\rho\nu(b)<1/2$.

Corollary 4: If $\rho\nu(b)\geq 1/2$, then the algorithm is unstable for all $\lambda>0$.

Corollary 4 is the intuitively expected generalization of the stability result obtained in the special case $\rho=1$ (Corollary 2). Also note that, as in the previous section, it can be shown that $P(\tau_i<\infty)=1$ for all $i\geq 0$ if and only if $\rho\nu(b)\leq 1/2$.

We now assume that $\rho\nu(b)<1/2$ and calculate the throughput of the algorithm. Since in this case the state of the channel in the last slot of each CRI (or sub-CRI) is not necessarily g, we calculate first the probability that the state of the channel in such a slot is f $(f\in E)$ given that in the first slot of the same CRI (or sub-CRI) the state of the channel is s $(s\in E)$ and n messages are transmitted. Denote this probability by $w_n^s(f)$. Since $\rho\nu(b)<1/2$ it follows that $w_n^s(b)=1-w_n^s(g)$ for all $n\geq 0$. We have $w_0^g(g)=w_1^g(g)=1$, and $w_0^b(g)=w_1^b(g)=w^b(g)$ where $w^b(g)$ is given by Proposition 2. For $n\geq 2$, we have

$$\begin{aligned} w_n^s(g)=\sum_{j=0}^{n}Q_n(j)\Big\{ & \big[r_sw_j^g(g)+(1-r_s)w_j^b(g)\big] \\ & \cdot\big[r_gw_{n-j}^g(g)+(1-r_g)w_{n-j}^b(g)\big] \\ & +\big[r_sw_j^g(b)+(1-r_s)w_j^b(b)\big] \\ & \cdot\big[r_bw_{n-j}^g(g)+(1-r_b)w_{n-j}^b(g)\big]\Big\}, \qquad s\in E. \end{aligned} \tag{5.8}$$

Equation (5.8) is explained in a way similar to (5.1) (here the strong Markov property of $\{X_n,Y_n,V_n),\ n\geq 0\}$ is used). Solving (5.8) for $(w_n^g(g),w_n^b(g))$ gives a recurrence equation for which the initial conditions are given above.

We now have

$$\begin{aligned} L_n^z=1+\sum_{j=0}^{n}Q_n(j)\Big\{ & r_z\big[L_j^g+\big[w_j^g(g)r_g+w_j^g(b)r_b\big]L_{n-j}^g \\ & +\big[w_j^g(g)(1-r_g)+w_j^g(b)(1-r_b)\big]L_{n-j}^b\big] \\ & +(1-r_z)\big[L_j^b+\big[w_j^b(g)r_g+w_j^b(b)r_b\big]L_{n-j}^g \\ & +\big[w_j^b(g)(1-r_g)+w_j^b(b)(1-r_b)\big]L_{n-j}^b\big]\Big\} \end{aligned} \tag{5.9}$$

for all $n\geq 2$ and $z\in E$. Equation (5.9) is explained in a way similar to (4.4) and (5.1). Solving (5.9) for (L_n^g,L_n^b) gives a recurrence equation for which the initial conditions are $L_0^g=L_1^g=1$, and $L_0^b=L_1^b=L^b$ where L^b is given by the first assertion of Theorem 2. $L(z)$ is now obtained as in (4.7).

Recall that T_i and τ_i, $i\geq 0$, are the first slot and the last slot of the ith CRI, respectively ($T_0=0$). Contrary to the case $\rho=1$, $\{X_{T_i},\ i\geq 1\}$ is not an i.i.d. process here.

Proposition 3: If $P(\tau_i<\infty)=1$ for all $i\geq 0$, then $\{X_{T_i},\ i\geq 1\}$ is a homogeneous Markov chain.

The proof of Proposition 3 is given in the Appendix.

Let the transition matrix of $\{X_{T_i},\ i\geq 1\}$ be

$$S=\begin{bmatrix} s_g & 1-s_g \\ s_b & 1-s_b \end{bmatrix}.$$

The transition probabilities s_g and s_b are computed as follows. Let $w^s(f)$, where $s,f\in E$, be the probability that the state of the channel in the last slot of a CRI is f given that the state of the channel in the first slot of the same CRI is s. It is clear that for $s\in E$,

$$w^s(g)=\sum_{n=0}^{\infty}\frac{e^{-\lambda\Delta}(\lambda\Delta)^n}{n!}w_n^s(g)\qquad w^s(b)=1-w^s(g).$$

We have

$$s_z=w^z(g)r_g+w^z(b)r_b,\qquad z\in E.$$

Let $\mu(b)$ and $\mu(g)$ be the invariant probabilities of $\{X_{T_i},$

$i \geq 1\}$. Then

$$\mu(g) = \frac{s_b}{s_b + 1 - s_g} = \frac{r_g w^b(g) + r_b w^b(b)}{(r_g - r_b)[w^b(g) - w^g(g)] + 1}$$

$$\mu(b) = 1 - \mu(g).$$

Now, the throughput of the algorithm is

$$T = \lim_{k\to\infty} \frac{E\left[\sum_{i=1}^{k} m_i\right]}{E\left[\sum_{i=1}^{k} l_i\right]} = \lim_{k\to\infty} \frac{\frac{1}{k}\sum_{i=1}^{k} E[m_i]}{\frac{1}{k}\sum_{i=1}^{k} E[l_i]}$$

$$= \lim_{k\to\infty} \frac{\frac{1}{k}\sum_{i=1}^{k} (\lambda\Delta)}{\frac{1}{k}\sum_{i=1}^{k} E[E(l_i|X_{T_i})]}$$

$$= \lim_{k\to\infty} \frac{\lambda\Delta}{\frac{1}{k}\sum_{i=1}^{k} E[L(X_{T_i})]} = \frac{\lambda\Delta}{\mu(g)L(g) + \mu(b)L(b)}$$

where the last equality follows from [14, p. 159]. Obviously, the results of this subsection for $\rho = 1$ coincide with the results of Section IV.

B. Modifications for the Case $\rho_0 \neq \rho_1$

In this case we have to distinguish between two types of error trees: an error tree that no message is transmitted in the slot that corresponds to its root (type 0) and an error tree that a single message is transmitted in the slot that corresponds to its root (type 1).

Any CRI that starts with an idle slot contains only idle slots and therefore the analysis for an error tree of type 0 is exactly the same as the analysis for the error tree in the previous subsection. Thus if $\rho, w^b(g), L^b, l_k$ in the previous subsection are replaced by $\rho_0, w_0^b(g), L_0^b, l_k^{(0)}$, respectively (where $l_k^{(0)}$ is the length of the sub-CRI that corresponds to the kth error tree of type 0), then Proposition 2 and Theorem 2 are true also in the case $\rho_0 \neq \rho_1$.

Assume now that $\rho_0 \nu(b) < 1/2$ and consider the kth error tree of type 1. By an equation analogous to (5.3), we obtain that the length of the sub-CRI that corresponds to this error tree is finite with probability 1. This implies that $w_1^b(b) = 1 - w_1^b(g)$. Using this we obtain in a way similar to (5.1) that $w_1^b(g)$ is the solution of the following first order equation:

$$\begin{aligned} w_1^b(g) = \rho_1\Big\{ q\big[[r_b + (1-r_b)w_1^b(g)][r_g + (1-r_g)w_0^b(g)] \\ + (1-r_b)(1-w_1^b(g))[r_b + (1-r_b)w_0^b(g)]\big] \\ + (1-q)\big[[r_b + (1-r_b)w_0^b(g)][r_g + (1-r_g)w_1^b(g)] \\ + (1-r_b)(1-w_0^b(g))[r_b + (1-r_b)w_1^b(g)]\big]\Big\} \quad (5.10) \end{aligned}$$

where $q \in (0,1)$ is the splitting parameter of the algorithm. An equation for L_1^b can be obtained in a way similar to (5.7), and solving this equation, we obtain that either $L_1^b = \infty$ or $L_1^b = 1 + V/W$ where

$$\begin{aligned} V \triangleq \rho_1\Big\{ q\big[r_b[1 + r_g + (1-r_g)L_0^b] \\ + (1-r_b)[1 + w_1^b(g)[r_g + (1-r_g)L_0^b] \\ + (1-w_1^b(g))[r_b + (1-r_b)L_0^b]]\big] \\ + (1-q)[2r_b + (1-r_b)(L_0^b + 1)]\Big\} \end{aligned}$$

$$\begin{aligned} W \triangleq 1 - \rho_1\big[(1-q)[r_b(1-r_g) + (1-r_b)[w_0^b(g)(1-r_g) \\ + (1-w_0^b(g))(1-r_b)]] + q(1-r_b)\big] \quad (5.11) \end{aligned}$$

(the second solution is valid if and only if $W \neq 0$). We have

$$\begin{aligned} W \geq 1 - \rho_1\big[(1-q)[r_b(1-r_g) + (1-r_b) \\ \cdot [w_0^b(g)\cdot 1 + (1-w_0^b(g))\cdot 1]] + q(1-r_b)\big] \\ = 1 - \rho_1[(1-q)(1-r_b r_g) + q(1-r_b)] \\ \geq 1 - \rho_1[(1-q)\cdot 1 + q\cdot 1] = 1 - \rho_1 \geq 0. \quad (5.12) \end{aligned}$$

Note that $\nu(b) = 1$ if and only if $r_b = 0$ and that $r_b = 0$ implies $w_0^b(g) = 0$. Therefore, equality holds in (5.12) if and only if $\rho_1 \nu(b) = 1$. Using this and the finiteness of L_0^b, we can obtain in a way similar to Theorem 1 that $L_1^b < \infty$ if and only if $\rho_1 \nu(b) < 1$, in which case $L_1^b = 1 + V/W$ where V and W are given by (5.11). Note that, by assumption (see Section III), r_g, $r_b \in (0,1)$ and $0 < \rho_1 \leq 1$, which imply that $\rho_1 \nu(b) < 1$.

It follows from these results that if ρ is replaced by ρ_0 then Corollaries 3 and 4 are true also in the case $\rho_0 \neq \rho_1$. The rest of the analysis in the previous subsection applies also, provided that we use for $w_1^b(g)$ and L_1^b the expressions given above.

VI. Summary and Discussion

We considered the problem of splitting algorithms in channels with errors when the channel has memory. We introduced a channel model described by a two-state Markov chain, and analyzed the operation of the tree CRA in this channel. We first obtained a stability result, i.e., the necessary conditions on the channel parameters for stability of the algorithm. Then, assuming that the stability conditions hold, we calculated the throughput of the algorithm.

The stability result we obtained generalizes the well-known result for memoryless channels (that the tree CRA is unstable if the probability that an error occurs in an idle slot is equal to or greater than 1/2).

The ideas and techniques we employed in this paper can be used to analyze other splitting algorithms as well. Moreover, they can be easily extended to apply to the analysis of the more general channel model in which the state space of the Markov chain contains an arbitrary finite number of states, where the probability of an error in state i is $\rho_o^{(i)}$ or $\rho_1^{(i)}$, according to whether no message or only a single message is transmitted in the slot. In fact, the analysis for this model differs from our analysis mainly in the number of equations required, which increases with the number of states.

Recently, it came to our attention that a similar problem was independently considered in [11].

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Appendix

Proof of Theorem 1

If $\nu(b)>1/2$, then $x_2<1$; hence $L^b=x_1=\infty$. If $\nu(b)=1/2$, then clearly $L^b=x_1=\infty$. If $\nu(b)<1/2$, then it suffices to show that the algorithmic parameter L^b must satisfy $L^b\le 1+[1/(1-r_g+r_b)/(1/2-\nu(b))]$, since this implies that $L^b=x_2=1+[1/(1-r_g+r_b)/(1/2-\nu(b))]$. To that end, for every $j\ge 1$ define $l_k(j)\triangleq l_k\wedge j$ (where $u\wedge v\triangleq\min(u,v)$) and $L^b(j)\triangleq E[l_k(j)]$. It follows immediately that $0<l_k(j)\uparrow l_k$ as $j\to\infty$; hence by the monotone convergence theorem $L^b(j)\uparrow L^b$ as $j\to\infty$. Let R_n be the length of the (sub-)CRI that starts at $t=n$. Denote $P_b(\cdot)=P(\cdot|X_{T_k}=b)$ and $E_b(\cdot)=E(\cdot|X_{T_k}=b)$. From the operation of the algorithm it follows that $R_{T_k}=1+R_{T_k+1}+R_{T_k'}$ and hence for all $j\ge 1$,

$$R_{T_k}\wedge j\le 1+R_{T_k+1}\wedge j+R_{T_k'}\wedge j. \tag{A.1}$$

We have

$$E_b(R_{T_k}\wedge j)=E(R_{T_k}\wedge j)=L^b(j),\qquad j\ge 1 \tag{A.2}$$

where the first equality follows from the definition of T_k and the second equality follows from the definition of $L^b(j)$. We also have

$$\begin{aligned}E_b(R_{T_k+1}\wedge j)&=E_b\big(E_b(R_{T_k+1}\wedge j|X_{T_k+1})\big)\\&=P_b(X_{T_k+1}=g)E_b(R_{T_k+1}\wedge j|X_{T_k+1}=g)\\&\quad+P_b(X_{T_k+1}=b)E_b(R_{T_k+1}\wedge j|X_{T_k+1}=b)\\&=r_b\cdot 1+(1-r_b)L^b(j),\qquad j\ge 1\end{aligned} \tag{A.3}$$

where the last equality follows from the operation of the algorithm and the strong Markov property of $\{X_n,\ n\ge 0\}$. By assumption, $\nu(b)<1/2$; hence by Proposition 1 we have $P(l_k<\infty)=1$. Using this, we obtain in a way similar to (A.3):

$$E_b(R_{T_k'}\wedge j)=E_b\big(E_b(R_{T_k'}\wedge j|X_{T_k'})\big)=r_g\cdot 1+(1-r_g)L^b(j),\qquad j\ge 1. \tag{A.4}$$

Taking expectations on both sides of (A.1) and using (A.2)–(A.4), we obtain that for all $j\ge 1$,

$$L^b(j)\le L^b(j)(2-r_g-r_b)+(r_g+r_b+1). \tag{A.5}$$

Now, since $\nu(b)<1/2$ and $L^b(j)<\infty$ for all $j\ge 1$, it follows from (A.5) that $L^b(j)\le 1+[1/(1-r_g+r_b)/(1/2-\nu(b))]$ for all $j\ge 1$. Letting $j\to\infty$, we obtain the result.

Proof of Lemma 1

a) Denote $h(x)=ax^2+bx+c(1)$ where $a,b,c(\cdot)$ are given by (5.2b). We have $h(0)=c(1)>0$. For $r_g>r_b$ the parabola $h(x)$ is concave and therefore there exist two real roots, x' and x'', which satisfy $x'<0<x''$. We have $h(1)=\rho-1\le 0$ for all $0<\rho\le 1$, and hence $x''\le 1$. For $r_b>r_g$ the parabola $h(x)$ is convex. For all $0<\rho<1$, we have $h(1)<0$ and hence there exist two real roots, x' and x'', which satisfy $0<x'<1<x''$. For $\rho=1$ we have $x'=1$ and $x''=c(1)/a=(r_b(1-r_b)+r_gr_b)/(r_b(1-r_b)-r_g(1-r_b))>1$. Thus part a) is proved for $y=1$. The result remains true when $0\le y<1$ since in this case $0<c(y)<c(1)$.

b) Denote $f(x)=\alpha x^2+\beta x+\gamma$ where α,β,γ are as in (5.5). We have $f(0)=\gamma>0$. For $r_g>r_b$ the parabola $f(x)$ is concave and therefore x_3 and x_4 given by (5.6) are real and satisfy $x4<0<x_3$. Thus x_4 is not in [0,1]. For $r_g<r_b$, it suffices to show that if $0\le y_4\le 1$, then x_4 is not in [0,1]. To that end, note that x_4 is a root of $ax^2+bx+c(y_4)=0$ where $a,b,c(\cdot)$ are given by (5.2b). By part a), for $0\le y_4\le 1$ the roots of this equation are real, thus x_4 is real. Hence x_3 is real too, and we have $x_3\le x_4$. Let x' and x'' be the roots of $ax^2+bx+c(y_4)=0$, and let $x'\le x''$. We have

$$x_4\ge -\frac{\beta}{2\alpha}=-\frac{b(1-r_b)-ar_b}{2a(1-r_b)}=-\frac{b}{2a}+\frac{r_b}{2(1-r_b)}>-\frac{b}{2a}\ge x'.$$

Hence $x_4=x''$ which by part a) is not in the interval [0,1].

Proof of Lemma 2

Consider first the case $r_g\ne r_b$. Since $P(l_k<\infty)=1$, it follows by (5.2) and Lemma 1-a) that $w^b(g)=x_1$. Substituting for x_1 the expression given by (5.6) yields $C(x_1)-A=F_1+G_1$ where $F_1\triangleq 1+b/2-\rho[1-r_br_g+(1-r_b)^2]$ and $G_1\triangleq\sqrt{b^2-4ac(1)}/2$. By straightforward calculations we obtain $(F_1+G_1)(F_1-G_1)=D[1/2-\rho\nu(b)]$ where $D=-2\rho(1-r_b)(1-r_g+r_b)<0$. Since $F_1+G_1\ge F_1-G_1$ it follows that if $\rho\nu(b)<1/2$, then $F_1-G_1<0$.

To show that $F_1-G_1<0$ also when $\rho\nu(b)\ge 1/2$, it suffices to show that if $\rho\nu(b)\ge 1/2$, then

$$-2F_1>2\rho(1-r_g)+r_g-r_b-1 \tag{A.6}$$

since $G_1\ge 0$ and $\rho\nu(b)\ge 1/2$ implies that $2\rho(1-r_g)+r_g-r_b-1\ge 0$. Substituting in (A.6) the expression given above for F_1, we obtain that (A.6) is equivalent to $\rho/(1-\rho)>(r_g-r_b)/(1-r_b)$ which holds for $\rho\nu(b)\ge 1/2$ (since then $\rho>1/2$).

Thus, $F_1-G_1<0$ and therefore

$$\begin{aligned}C(x_1)-A=F_1+G_1&=\frac{D}{F_1-G_1}[1/2-\rho\nu(b)]\\&=H[1/2-\rho\nu(b)]\end{aligned}$$

and clearly $H>0$.

In the case $r_g=r_b=r, C(\cdot)$ in (5.4) is a constant, and we immediately have $C-A=1-2\rho(1-r)=1-2\rho\nu(b)=2[1/2-\rho\nu(b)]$.

Proof of Proposition 3

For every $n\ge 0$ let $Z_n=(X_n,Y_n,V_n)$. For all $i\ge 0, T_i$ is a stopping time for $\{Z_n,\ n\ge 0\}$, and clearly $P(T_i<\infty)=1$ for all $i\ge 0$. From the operation of the algorithm it follows that $\{T_{i+1}=T_i+n\}\in\mathscr{F}(Z_{T_i},Z_{T_i+1},\cdots,Z_{T_i+n})$, and also that $T_{i+1}>T_i$ for all $i\ge 0$ hence $\mathscr{F}_{T_i}\subset\mathscr{F}_{T_{i+1}}$ for all $i\ge 0$. This implies that for any integer $i \geq 0$ and any $k_0, \cdots, k_{i+1} \in E$,

$$\{ X_{T_0} = k_0, \cdots, X_{T_{i-1}} = k_{i-1} \} \in \mathscr{F}_{T_i} \qquad \{ X_{T_{i+1}} = k_{i+1} \} \in \mathscr{F}'_{T_i} \tag{A.7}$$

where $\mathscr{F}'_{T_i}$ is the σ-algebra generated by the process $\{ Z_k, k \geq T_i \}$. Thus

$$\begin{aligned} P(X_{T_{i+1}} &= k_{i+1}, X_{T_i} = k_i, \cdots, X_{T_0} = k_0) \\ &= \sum_{y,v} P(X_{T_{i+1}} = k_{i+1}, X_{T_i} = k_i, \cdots, X_{T_0} = k_0, \\ &\qquad Y_{T_i} = y, V_{T_i} = v) \\ &= \sum_{y,v} P(X_{T_{i+1}} = k_{i+1} | X_{T_i} = k_i, Y_{T_i} = y, V_{T_i} = v) \\ &\qquad \cdot P(X_{T_i} = k_i, Y_{T_i} = y, V_{T_i} = v, X_{T_{i-1}} = k_{i-1}, \cdots, X_{T_0} = k_0) \\ &= \sum_{y,v} P(X_{T_{i+1}} = k_{i+1}, Y_{T_i} = y, V_{T_i} = v | X_{T_i} = k_i) \\ &\qquad \cdot P(X_{T_i} = k_i, \cdots, X_{T_0} = k_0) \\ &= P(X_{T_{i+1}} = k_{i+1} | X_{T_i} = k_i) \cdot P(X_{T_i} = k_i, \cdots, X_{T_0} = k_0) \end{aligned}$$

for any integer $i \geq 0$ and any $k_0, \cdots, k_{i+1} \in E$. The second equality follows from (A.7) and the strong Markov property of $\{ Z_n, n \geq 0\}$. The third equality follows from the fact that (Y_{T_i}, V_{T_i}) and $(X_{T_0}, \cdots, X_{T_i})$ are independent, which is implied by the Poisson arrivals and the definition of $\{V_n, n \geq 0\}$. Thus $\{ X_{T_i}, i \geq 0\}$ is a Markov chain.

The time-homogeneity of $\{ X_{T_i}, i \geq 0\}$ follows from the fact that, for any integer $i \geq 0$, the transition probabilities of the Markov chain $\{ Z_{T_i+n}, n \geq 0\}$, the function $f_i(\cdot)$ in the equality $1_{\{T_{i+1} = T_i + n\}} = f_i(Z_{T_i}, Z_{T_i+1}, \cdots, Z_{T_i+n})$, and the distribution of (Y_{T_i}, V_{T_i}) given X_{T_i} are all independent of i.

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