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MESSAGE DELAY DISTRIBUTION IN GENERALIZED TIME DIVISION MULTIPLE ACCESS (TDMA)

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In the classical TDMA, which has been the subject of extensive research in the past, a single slot is allocated to each and every user within every frame. In many situations this leads to an inefficient utilization of the channel calling for a nonhomogeneous allocation of slots. This paper focuses on a generalized TDMA scheme in which users are allocated more than a single slot per frame, not necessarily contiguous. We derive the delay distribution and buffer occupancy for such a scheme as well as the expected values of these quantities.

1. INTRODUCTION

High-speed multiaccess communication channels often operate under the time division multiple access (TDMA) scheme of channel sharing. Under the TDMA access scheme, time is divided into equally sized slots, preassigned to the different users. Every user is allowed to transmit freely during a slot assigned to it, that is, during an assigned slot the entire system resources are devoted to that user. The slot assignments follow a predetermined pattern that repeats itself periodically; each such period is called a cycle or a frame.

In the *basic* and most common TDMA scheme every user has exactly one slot assigned in every frame in which a single packet can be transmitted, result-

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ing in a system that operates in a round robin mode. Queueing models of this scheme with various variations have been introduced and analyzed in numerous papers [1–13]. The basic scheme operates reasonably well if the set of users is relatively homogeneous with respect to the traffic they generate. When this is not the case, it is possible to assign more than a single slot to some users in a manner that reflects their traffic demands. A *generalized* TDMA scheme is a scheme in which a user is allocated more than a single slot within a frame. Generalized TDMA schemes in which all slots dedicated to a user are contiguous in a frame have been analyzed in Hayes [14], Ko and Davis [15], and Bruneel [16]. These works consider a gated ("please wait") transmission strategy, namely, transmission of messages that arrive during any frame can start only in the subsequent frame. In Ko and Davis [15] and Bruneel [16] delay analysis is presented only for single packets, a unit of data that requires a single slot for its transmission, while Hayes [14] deals with messages, a unit of data that requires more than a single slot for its transmission, but does not analyze the delay.

This paper analyzes the performance of a generalized TDMA scheme with a given, yet arbitrary allocation of slots for any user. A flexible, nongated, transmission strategy is accommodated, namely, the transmission of a message can start in the first allocated slot after message arrival; this corresponds to the "come right in" transmission strategy of Hayes [14]. For this scheme several quantities are derived such as the message delay distribution and expected message delay, the steady-state distribution (generating function) of the number of packets in the user buffer, and their expected values (the latter quantities have been similarly derived in Anderson et al. [17] and using different techniques in Hofri and Rosberg [18]). Finally, in a discussion section, we dwell on proper ways of allocating the slots to improve performance.

2. THE QUEUEING MODEL

Consider a TDMA system in which a number of users share the common channel according to the TDMA discipline with several modifications that we specify. Because the performance of users is independent of one another, we focus our attention on one such user. *Messages* arrive to a user according to a Poisson process at a rate of λ messages per second, and every message consists of a random number $\tilde{L}$ of *packets*. Let $L(z)$ be the generating function of $\tilde{L}$, L its mean, and $\overline{L^2}$ its second moment. These packets, upon their arrival, are stored in a buffer that can accommodate an infinite number of packets.

Remark: Poisson arrival process is assumed for convenience of presentation only; analysis for any i.i.d. arrival process can be carried out exactly along the same lines.

The time axis $[0,\infty)$ is divided into intervals $[0,T_c),[T_c,2T_c),\ldots$ called frames. Each frame is further divided into N equal length subintervals called slots. The length of a slot is $T = T_c/N$ and the transmission time of a packet is exactly one slot. In the generalized TDMA scheme considered here a user is allocated more than one slot within a frame, with arbitrary distances between successive allocated slots. The allocation is constant and does not change from frame to frame. For reference we number the frames consecutively; the jth frame is the one between $(j-1)T_c$ and jT_c. Consider a user with K $(K < N)$ allocated slots and let $d(k) \geq 1$ $(1 \leq k \leq K)$ be the distance between allocation $(k+1) \bmod K$ and allocation $k \bmod K$ (we refer to this distance as the kth period). Notice that $\sum_{k=1}^{K} d(k) = T_c$. If its buffer is not empty the user transmits a single packet in every slot allocated to it. Thus, the first packet of a message arriving to a user with an empty buffer will be transmitted in the first subsequent slot allocated to the user. Messages are transmitted according to a first-come first-served policy. Without loss of generality we assume that the first slot in a frame belongs to the user under consideration.

Note that for buffer occupancy calculation the assumption that the message arrival process is Poisson is not essential. The same analysis applies for an arrival process that is independent between every two periods but is otherwise arbitrary.

3. ANALYSIS

A. Probability Distribution of Number of Packets at Slot Allocations

Let $\tilde{q}_j(k)$ be the number of packets awaiting transmission at the beginning of interval $d(k)$ $(1 \leq k \leq K)$ in the $j+1$st frame. We start by determining the generating function of the steady-state distribution of $\tilde{q}_j(k)$ $(1 \leq k \leq K)$. Steady-state distribution exists when $\rho \triangleq \lambda L T_c/K < 1$.

From the operation of generalized TDMA we have that the number of packets awaiting transmission at the beginning of interval $d(k+1)$ $(1 \leq k \leq K-1)$ in the $j+1$st frame equals the number of packets awaiting transmission at the beginning of interval $d(k)$, less the packet (if there were any) that has been transmitted in the kth allocated slot, plus the packets that arrived during $d(k)$. In addition, the number of packets awaiting transmission at the beginning of interval $d(1)$ in the $j+1$st frame equals the number of packets awaiting transmission at the beginning of interval $d(K)$ in the jth frame, less the packet (if there were any) that has been transmitted in the Kth allocated slot, plus the packets that arrived during $d(K)$. Therefore,

$$\begin{aligned} \tilde{q}_j(k+1) &= \tilde{q}_j(k) - \Delta_{\tilde{q}_j(k)} + \tilde{\nu}(k) \qquad 1 \leq k \leq K-1 \\ \tilde{q}_{j+1}(1) &= \tilde{q}_j(K) - \Delta_{\tilde{q}_j(K)} + \tilde{\nu}(K), \end{aligned} \tag{1}$$

where $\tilde{\nu}(k)$ $(1 \leq k \leq K)$ is the number of packets arriving to the user during $d(k)$, and $\Delta_{\tilde{q}} = 0$ if $\tilde{q} = 0$, $\Delta_{\tilde{q}} = 1$ if $\tilde{q} > 0$. Notice that $\tilde{\nu}(k)$ does not depend on j.

Let $\tilde{q}(k)$ be the number of packets awaiting transmission at the beginning of interval $d(k)$ $(1 \le k \le K)$ in steady-state and let $Q_k(z) = \sum_{i=0}^{\infty} \Pr[\tilde{q}(k) = i] z^i$ be the generating function of $\tilde{q}(k)$. Then from (1) we obtain after some algebra

$$Q_1(z) = \frac{(z-1) \sum_{u=1}^{K} Q_u(0) z^{u-1} \prod_{m=u}^{K} V_m(z)}{z^K - \prod_{m=1}^{K} V_m(z)} \tag{2}$$

$$Q_k(z) = Q_1(z) z^{-(k-1)} \prod_{m=1}^{k-1} V_m(z) + (1 - z^{-1}) \sum_{\nu=1}^{k-1} Q_\nu(0) z^{-(k-\nu-1)} \prod_{m=\nu}^{k-1} V_m(z) \qquad 2 \le k \le K \tag{3}$$

where $V_k(z) = E[z^{\tilde{\nu}(k)}] = e^{-\lambda d(k)[1-L(z)]}$ $(1 \le k \le K)$.

Had the boundary probabilities $Q_k(0) = \Pr[\tilde{q}(k) = 0]$ $(1 \le k \le K)$ been known, the generating functions $Q_k(z)$ $(1 \le k \le K)$ would be completely determined. To compute these probabilities we use a standard method. Consider the zeros of the denominator of (2) within the unit disk. Any such zero, $|z_n| \le 1$, satisfies the equation

$$z_n^K = \prod_{m=1}^{K} V_m(z_n) = e^{-\lambda T_c[1-L(z_n)]}. \tag{4}$$

When the stability condition holds ($\rho < 1$), (4) has exactly K roots within the unit disk, and all are distinct [14]. One of these roots is $z_K = 1$. The other roots are denoted by $z_1, z_2, \ldots, z_{K-1}$. Since $Q_1(z)$ is analytic within the unit disk, the numerator of (2) must vanish whenever the denominator vanishes within the unit disk. We thus substitute the values of z_n $(1 \le n \le K-1)$ into the numerator of (2) and obtain the following $K-1$ equations:

$$\sum_{\nu=1}^{K} Q_\nu(0) z_n^{\nu-1} \prod_{m=\nu}^{K} V_m(z_n) = 0 \qquad 1 \le n \le K-1. \tag{5}$$

An additional equation comes from the normalization condition $Q_1(z)|_{z=1} = 1$, namely (we use L'Hospital's rule)

$$K - \lambda T_c L = \sum_{\nu=1}^{K} Q_\nu(0). \tag{6}$$

It is not difficult to verify that the set of K equations (5)–(6) has a solution, completing the determination of $Q_k(0)$ $(1 \le k \le K)$ [14].

To summarize, the actual solution procedure is finding the roots of Eq. (4) within the unit disk and then solving the set of Eqs. (5) and (6). The solutions are then substituted into Eq. (2). Solving (4) is, by all counts, the toughest part of the procedure. One quite efficient method to do it is due to Mueller [19,20]. This method is particularly useful since it is iterative, does not require the evaluation of derivatives, obtains both real and complex roots even when these are not simple, and converges almost quadratically in the vicinity of a root. Another alternative for computing the boundary probabilities is to use Neuts' theory of matrix geometric computation [21].

Having computed the generating function of the number of packets at slot allocations, computing the generating function for the number of packets at the beginning of an arbitrary slot is straightforward.

B. Expected Number of Packets at the Beginning of an Allocated Slot

The expected number of packets at the beginning of an allocated slot in steady-state can be computed by evaluating the derivative of $Q_k(z)$ with respect to z at $z = 1$ [see (3) and (2)]. An alternative method (the one we employ here) is to use (1) directly. To that end, we square, take expectations of both sides of (1) and let $j \to \infty$. We obtain

$$\begin{aligned} E[\tilde{q}^2(k+1)] &= E[\tilde{q}^2(k)] + E[\Delta^2_{\tilde{q}(k)}] + E[\tilde{\nu}^2(k)] + 2E[\tilde{q}(k)\tilde{\nu}(k)] \\ &\quad - 2E[\tilde{q}(k)\Delta_{\tilde{q}(k)}] - 2E[\Delta_{\tilde{q}(k)}\tilde{\nu}(k)] \qquad 1 \le k \le K-1 \\ E[\tilde{q}^2(1)] &= E[\tilde{q}^2(K)] + E[\Delta^2_{\tilde{q}(K)}] + E[\tilde{\nu}^2(K)] + 2E[\tilde{q}(K)\tilde{\nu}(K)] \\ &\quad - 2E[\tilde{q}(K)\Delta_{\tilde{q}(K)}] - 2E[\Delta_{\tilde{q}(K)}\tilde{\nu}(K)]. \end{aligned} \tag{7}$$

Let $q(k) \triangleq E[\tilde{q}(k)]$ and $\nu(k) \triangleq E[\tilde{\nu}(k)]$ for $1 \le k \le K$. With these notations and using the independence between $\tilde{q}$ and $\tilde{\nu}$ and the identities $E[\Delta^2_{\tilde{q}(k)}] = E[\Delta_{\tilde{q}(k)}] = 1 - Q_k(0)$; $E[\tilde{q}(k)\Delta_{\tilde{q}(k)}] = E[\tilde{q}(k)]$, we obtain

$$\begin{aligned} q^2(k+1) &= q^2(k) + E[\tilde{\nu}^2(k)] + [1 - Q_k(0)][1 - 2\nu(k)] \\ &\quad - 2q(k)[1 - \nu(k)] \qquad 1 \le k \le K-1 \end{aligned} \tag{8a}$$

$$\begin{aligned} q^2(1) &= q^2(K) + E[\tilde{\nu}^2(K)] + [1 - Q_K(0)][1 - 2\nu(K)] \\ &\quad - 2q(K)[1 - \nu(K)]. \end{aligned} \tag{8b}$$

Summing (8b) and (8a) for all $k = 1,2,\ldots,K-1$, we obtain

$$2\sum_{k=1}^{K} q(k)[1 - \nu(k)] = \sum_{k=1}^{K} E[\tilde{\nu}^2(k)] + \sum_{k=1}^{K} [1 - Q_k(0)][1 - 2\nu(k)]. \tag{9}$$

Using (1) we have

$$\sum_{k=1}^{K} q(k)\,[1-\nu(k)] = q(1)\left(K - \sum_{k=1}^{K} \nu(k)\right) + \sum_{k=1}^{K} [1-\nu(k)] \sum_{m=1}^{k-1} (\nu(m) - E[\Delta_{\tilde{q}(m)}]) \qquad \textbf{(10)}$$

where an empty sum vanishes. Substituting (10) into (9) we obtain

$$q(1) = \frac{\sum_{k=1}^{K} E[\tilde{\nu}^2(k)] + \sum_{k=1}^{K} [1-Q_k(0)]\,[1-2\nu(k)]}{2\left[K - \sum_{k=1}^{K} \nu(k)\right]} - \frac{2\sum_{k=1}^{K} [1-\nu(k)] + \sum_{m=1}^{k-1} (\nu(m) - E[\Delta_{\tilde{q}(m)}])}{2\left[K - \sum_{k=1}^{K} \nu(k)\right]}. \qquad \textbf{(11)}$$

Because the arrival process is Poisson we have $\nu(k) = \lambda L d(k)$ and $E[\tilde{\nu}^2(k)] = \lambda \overline{L^2} d(k) + \lambda^2 L^2 d^2(k)$. Also, from (6) $\sum_{k=1}^{K} [1-Q_k(0)] = \lambda L T_c$. Therefore, we obtain from (11),

$$q(1) = \frac{\lambda L T_c + \lambda \sum_{k=1}^{K} d(k)\left[\overline{L^2} + \lambda L^2 d(k) - 2L[1-Q_k(0)]\right]}{2(K-\lambda L T_c)} - \frac{2\sum_{k=1}^{K} [1-\lambda L d(k)] \sum_{m=1}^{k-1} [\lambda L d(m) - 1 + Q_m(0)]}{2(K-\lambda L T_c)} \qquad \textbf{(12)}$$

and finally from (1) we get

$$\begin{aligned} q(k) &= q(1) + \sum_{m=1}^{k-1} [\nu(m) - E[\Delta_{\tilde{q}(m)}]] \\ &= q(1) + \sum_{m=1}^{k-1} [\lambda L d(m) - 1 + Q_m(0)] \qquad 2 \le k \le K. \end{aligned} \qquad \textbf{(13)}$$

C. Message Delay Distribution

Consider a tagged message within the kth period $d(k)$ $(1 \le k \le K)$, $\tilde{w}(k)$ seconds before the end of the interval $d(k)$ and tag its last packet. The delay of the tagged message is the time elapsed from its arrival, until its last packet is transmitted, that is, it is the delay of the tagged packet. Denote this delay by $\tilde{D}$. Thus, if $\tilde{I}(k)$ is a random variable representing the total number of packets that are to be transmitted before the tagged packet, then the delay of the tagged packet is $\tilde{w}(k)$ plus the time needed to transmit the $\tilde{I}(k)$ packets plus the time to transmit the tagged packet itself [note that $\tilde{I}(k)$ depends on $\tilde{w}(k)$].

Some insight into $\tilde{I}(k)$ is appropriate. The quantity $\tilde{I}(k)$ is actually the number of intervals that have to elapse before the interval in which the tagged packet is transmitted. This number can be (uniquely) decomposed into a number of complete frames and some left over. In other words, we can write

$$\tilde{I}(k) = \tilde{f}(k) \cdot K + \tilde{J}(k) \qquad 0 \leq \tilde{J}(k) \leq K - 1 \tag{14}$$

where $\tilde{f}$ designates the number of complete frames of delay and $\tilde{J}$ designates the number of intervals left over. Both $\tilde{J}(k)$ and $\tilde{f}(k)$ are non-negative integer-valued random variables and their distributions are derived in Appendix A.

The delay of the tagged message after waiting the initial $\tilde{w}(k)$ seconds is $\tilde{f}(k)T_c$ seconds (representing the number of complete frames) plus the time to transmit the $\tilde{J}(k)$ leftover packets which requires $\sum_{u=k+1}^{k+\tilde{J}(k)} d(u)$ seconds if there are any packets left. In all cases, the transmission time of the tagged packet is T. In summary, given $\tilde{w}(k)$, the total delay is

$$\tilde{D}[k \mid \tilde{w}(k), \tilde{I}(k)] = \begin{cases} \tilde{w}(k) + \tilde{f}(k)T_c + T & \tilde{J}(k) = 0 \\ \tilde{w}(k) + \tilde{f}(k)T_c + \sum\limits_{u=k+1}^{k+\tilde{J}(k)} d(u) + T & 1 \leq \tilde{J}(k) \leq K - 1 \end{cases}$$

where the summation wraps around from K to 1 when necessary. The above can be rewritten (along with the relation $\tilde{f}K = \tilde{I} - \tilde{J}$) as follows:

$$\tilde{D}[k \mid \tilde{w}(k), \tilde{I}(k)] = \tilde{w}(k) + T + \frac{T_c}{K}\tilde{I}(k) - \frac{T_c}{K}\tilde{J}(k) + \sum_{u=k+1}^{k+1+\tilde{J}(k)} d(u) - d[k + 1 + \tilde{J}(k)]. \tag{15}$$

For notational convenience, let us define

$$A(j,k) \triangleq \sum_{u=k+1}^{k+1+j(k)} d(u) - d[k + 1 + j(k)]$$

which then turns Eq. (15) into

$$\tilde{D}[k\,|\,\tilde{w}(k),\tilde{I}(k)] = \tilde{w}(k) + T + \frac{T_c}{K}\,\tilde{I}(k) - \frac{T_c}{K}\,\tilde{J}(k) + A(\tilde{J},k). \tag{16}$$

Note that $\tilde{I}$, and hence $\tilde{J}$ depend on $\tilde{w}(k)$. Moving to the Laplace transform domain, we define

$$D_k^*[s\,|\,\tilde{w}(k),\tilde{I}(k)] \triangleq E[e^{-s\tilde{D}[k|\tilde{w}(k),\tilde{I}(k)]}]$$

and gradually eliminating the conditions (the details of which are given in Appendix B), we obtain

$$\begin{aligned} D^*(s) &= \sum_{k=1}^{K} \frac{d(k)}{T_c}\, D_k^*(s) \\ &= \frac{e^{-sT}}{K} \sum_{k=1}^{K} \sum_{m=0}^{K-1} Q_{k+1}(z_s\alpha_m) L(z_s\alpha_m)\, \frac{1 - e^{-d(k)\{s-\lambda[1-L(z_s\alpha_m)]\}}}{d(k) z_s\alpha_m\{s - \lambda[1 - L(z_s\alpha_m)]\}} \\ &\quad \times \left[\sum_{j=0}^{K-1} e^{-sA(j,k)} (z_s\alpha_m)^{-j}\right] \end{aligned} \tag{17}$$

where $\alpha_m = e^{i(2\pi m/K)}$ is the unit root of order K and $z_s \triangleq e^{-s(T_c/K)}$. This expression is the Laplace transform of the message delay distribution.

D. Expected Delay of a Message

The expected delay of a message can be computed by evaluating the derivative of $D^*(s)$ with respect to s at $s = 0$ [see (17)]. An alternative method (the one we employ here) is to use (16) directly. To that end, we take expectation on (16) and obtain,

$$D(k) = E[\tilde{w}(k)] + T + \frac{T_c}{K}\, E[\tilde{I}(k)] - \frac{T_c}{K}\, E[\tilde{J}(k)] + E[A(\tilde{J},k)]. \tag{18}$$

We now compute each of the terms in (18). Clearly,

$$E[\tilde{w}(k)] = \frac{1}{2}\, d(k). \tag{19}$$

From (13),

$$\begin{aligned} E[\tilde{I}(k)] &= q(k) - [1 - Q_k(0)] + \frac{1}{2}\,\lambda L d(k) + L - 1 \\ &= q(1) + L - 1 - \frac{1}{2}\,\lambda L d(k) + \sum_{m=1}^{k} [\lambda L d(m) - 1 + Q_m(0)]. \end{aligned} \tag{20}$$

Using the distribution of $\tilde{J}$ (from Appendix A) and the definition of $h_k(z)$ we have

$$E[\tilde{J}(k)] = E[E[\tilde{J}(k)|\,\tilde{w}(k)]] = E\left[\frac{K-1}{2} - \sum_{m=1}^{K-1} \frac{l_k(\alpha_m, \tilde{w}(k))}{1 - \dfrac{1}{\alpha_m}}\right]$$

$$= \frac{K-1}{2} - \sum_{m=1}^{K-1} \frac{E[l_k(\alpha_m, \tilde{w}(k))]}{1 - \dfrac{1}{\alpha_m}}$$

$$= \frac{K-1}{2} - \sum_{m=1}^{K-1} \frac{h_k(\alpha_m)}{1 - \dfrac{1}{\alpha_m}} E[e^{a_m \tilde{w}(k)}] \tag{21}$$

$$= \frac{K-1}{2} - \sum_{m=1}^{K-1} \frac{h_k(\alpha_m)}{1 - \dfrac{1}{\alpha_m}} \frac{1}{d(k)} \int_0^{d(k)} e^{a_m w} dw$$

$$= \frac{K-1}{2} - \sum_{m=1}^{K-1} \frac{h_k(\alpha_m)}{1 - \dfrac{1}{\alpha_m}} \frac{e^{a_m d(k)} - 1}{a_m d(k)}$$

where $a_m \triangleq \lambda[1 - L(\alpha_m)]$.

In a similar manner,

$$E[A(\tilde{J}, k)] = E[E[A(\tilde{J}, k)|\,\tilde{w}(k)]] = E\left[\sum_{j=0}^{K} A(j,k)\Pr[\tilde{J} = j\,|\,\tilde{w}(k)]\right]$$

$$= E\left[\frac{1}{K}\sum_{j=0}^{K-1} A(j,k) \sum_{m=0}^{K-1} \alpha_m^{-j} l_k(\alpha_m, \tilde{w}(k))\right]$$

$$= \frac{1}{K}\sum_{j=0}^{K-1}\sum_{m=0}^{K-1} A(j,k)\alpha_m^{-j} h_k(\alpha_m) E[e^{a_m \tilde{w}(k)}].$$

Considering that $A(j,k) = 0$ for $j = 0$, $\alpha_0 = 1$, and $a_0 = 0$, the above yields

$$E[A(\tilde{J}, k)] = \frac{1}{K}\sum_{j=1}^{K-1} A(j,k) + \frac{1}{K}\sum_{m=1}^{K-1}\sum_{j=1}^{K-1} A(j,k)\alpha_m^{-j} h_k(\alpha_m) \frac{e^{a_m d(k)} - 1}{a_m d(k)}. \tag{22}$$

Combining all terms, we obtain

$$
\begin{aligned}
D(k) &= \frac{1}{2}\, d(k) + T \\
&+ \frac{T_c}{K}\left(q(1) + L - 1 - \frac{1}{2}\,\lambda L d(k) + \sum_{m=1}^{k} \left[\lambda L d(m) - 1 + Q_m(0)\right]\right) \\
&- \frac{T_c(K-1)}{2K} + \frac{1}{K}\sum_{j=1}^{K-1} A(j,k) \\
&+ \frac{1}{Kd(k)} \sum_{m=1}^{K-1} h_k(\alpha_m)\, \frac{e^{a_m d(k)} - 1}{a_m} \left(\frac{T_c}{1 - \dfrac{1}{\alpha_m}} + \sum_{j=1}^{K-1} A(j,k)\alpha_m^{-j} \right).
\end{aligned} \tag{23}
$$

Finally, the expected delay of a message is

$$
D = \sum_{k=1}^{K} \frac{d(k)}{T_c}\, D(k). \tag{24}
$$

4. DISCUSSION

A natural question to ask is how to allocate the K slots available to a user in a frame, in order to improve the performance. When the expected number of packets in the user's buffer is the performance measure, or equivalently, when expected *packet* delay is the measure, then it was shown [18] that the best allocation is the uniform one, namely, all the internal periods $d(k)$ $(1 \le k \le K)$ should be equal. Furthermore, this allocation remains optimal for all arrival rates.

When the expected *message* delay is used as the performance measure, we realized, via many numerical examples, that the optimal allocation depends both on the arrival rate λ and on the specific distribution of the message length. Whereas complete characterization of the optimal allocation pattern is still an open question, the following captures some of our observations.

When a message arrives at the user's buffer, its delay is affected by the number of packets ahead of it in the buffer. This number amounts to a number of whole frames plus some left over; the allocation of slots within a frame can affect only this leftover. Thus, for heavy load ($\rho \to 1$), the expected message delay is not very sensitive to changes in the interallocation distances since the major portion of the delay is due to the large number of whole frames a message must wait before its transmission starts. For light load ($\rho \to 0$ or equivalently $\lambda \to 0$), the expected message delay is very sensitive to the allocation distances, and it is given by

$$
D_{\lambda \to 0} = 1 + \frac{1}{T_c} \sum_{i=1}^{K} \gamma_i \left[\frac{1}{2} \sum_{j=1}^{K} d^2(j) + \sum_{j=1}^{K} \sum_{l=j+1}^{j+i-1} d(j) d(l) \right] \tag{25}
$$

where γ_i ($1 \leq i \leq K$) is the probability that a message transmission requires i slots beyond the number of whole frames, or in other words, the probability that a message length is i mod K (clearly $\sum_{i=1}^{K} \gamma_i = 1$). After some algebra (25) is transformed to

$$D_{\lambda \to 0} = 1 + \frac{1}{2} T_c + \frac{1}{T_c} \sum_{l=1}^{K} \sum_{k=1}^{K-l} d(l)d(l+k) \sum_{i=k+1}^{K} (\gamma_i - \gamma_{K-i+1}). \qquad \textbf{(26)}$$

From (26) we observe that if $\gamma_i = \gamma_{K-i+1}$ for $i = 1,2,\ldots,K$, then the expected delay is completely independent of the interallocation distances. Another special case, assume $\gamma_i = \gamma_{K-i+1}$ for $i = 2,3,\ldots,K-1$, then it follows from (26) that for $\gamma_1 > \gamma_K$ the optimal allocation is the uniform (equidistant) one while for $\gamma_1 < \gamma_K$ the K slots should be contiguous in order to minimize $D_{\lambda \to 0}$.

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APPENDIX A

DISTRIBUTION OF THE MOD FUNCTION

Let $\tilde{l}$ be a non-negative integer valued random variable with a known distribution and a generating function $l(z)$, and let K be a known integer constant. The quantity $\tilde{l}$ can be uniquely decomposed into

$$\tilde{l} = \tilde{f} \cdot K + \tilde{J} \qquad 0 \le \tilde{J} \le K - 1. \tag{27}$$

In another form this can be written as $\tilde{J} = \tilde{l} \bmod K$ and $\tilde{f} = \lfloor \tilde{l}/K \rfloor = (\tilde{l} - \tilde{J})/K$. We would like to compute the distributions of $\tilde{J}$ and $\tilde{f}$ from that of $\tilde{l}$.

Let α_m be the unit roots of order K, namely, $\alpha_m = e^{j(2\pi m/K)}$. These roots obey

$$\frac{1}{K}\sum_{m=0}^{K-1} \alpha_m^n = 1 - \Delta_{n \bmod K} = \begin{cases} 1 & K \text{ divides } n \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

Our most basic relation is derived as follows:

$$\begin{aligned} \frac{1}{K}\sum_{m=0}^{K-1} (z\alpha_m)^{-n} l(z\alpha_m) &= \frac{1}{K}\sum_{m=0}^{K-1} (z\alpha_m)^{-n} \sum_{l=0}^{\infty} \Pr[\tilde{l} = l]\,(z\alpha_m)^l \\ &= \frac{1}{K}\sum_{l=0}^{\infty} \Pr[\tilde{l} = l] z^{l-n} \sum_{m=0}^{K-1} (\alpha_m)^{l-n} \\ &= \sum_{l=0}^{\infty} \Pr[\tilde{l} = l] z^{l-n} (1 - \Delta_{(l-n) \bmod K}) \\ &= \sum_{f=0}^{\infty} \Pr[\tilde{l} = fK + n] z^{fK}. \end{aligned} \tag{29}$$

By setting $z = 1$ in Eq. (29) we get

$$\Pr[\tilde{J} = n] = \sum_{f=0}^{\infty} \Pr[\tilde{l} = Kf + n] = \frac{1}{K}\sum_{m=0}^{K-1} \alpha_m^{-n} l(\alpha_m) \qquad 0 \le n \le K - 1 \tag{30}$$

which gives us the distribution of $\tilde{J}$. From this $J(z)$, the generating function of $\tilde{J}$, can be computed as follows:

$$\begin{aligned} \sum_{n=0}^{K-1} \Pr[\tilde{J} = n] z^n &= \sum_{n=0}^{K-1} \left[\frac{1}{K}\sum_{m=0}^{K-1} \alpha_m^{-n} l(\alpha_m)\right] z^n = \frac{1}{K}\sum_{m=0}^{K-1} \left[\sum_{n=0}^{K-1} \left(\frac{z}{\alpha_m}\right)^n\right] l(\alpha_m) \\ &= \frac{1}{K}\sum_{m=0}^{K-1} \frac{1 - \left(\dfrac{z}{\alpha_m}\right)^K}{1 - \dfrac{z}{\alpha_m}} l(\alpha_m) = \frac{1}{K}\sum_{m=0}^{K-1} \frac{1 - z^K}{1 - \dfrac{z}{\alpha_m}} l(\alpha_m) \\ &= \frac{1 - z^K}{K} \sum_{m=0}^{K-1} \frac{l(\alpha_m)}{1 - \dfrac{z}{\alpha_m}} \end{aligned}$$

where in the step before last we used the fact that $\alpha_m^K = 1$. Overall, we thus have

$$J(z) = \frac{1-z^K}{K}\sum_{m=0}^{K-1}\frac{l(\alpha_m)}{1-\dfrac{z}{\alpha_m}} = \frac{1-z^K}{K(1-z)} + \frac{1-z^K}{K}\sum_{m=1}^{K-1}\frac{l(\alpha_m)}{1-\dfrac{z}{\alpha_m}}. \qquad (31)$$

Taking the derivative at $z = 1$ yields the expectation

$$E[\tilde{J}] = \frac{K-1}{2} - \sum_{m=1}^{K-1}\frac{l(\alpha_m)}{1-\dfrac{1}{\alpha_m}}. \qquad (32)$$

We turn now to calculate the generating function of $\tilde{f}$. Clearly

$$\Pr[\tilde{f}=f] = \sum_{n=0}^{K-1}\Pr[\tilde{l}=fK+n] \qquad f \geq 0$$

and thus

$$F(z) = \sum_{f=0}^{\infty}\Pr[\tilde{f}=f]z^f = \sum_{f=0}^{\infty}\left[\sum_{n=0}^{K-1}\Pr[\tilde{l}=fK+n]\right]z^f = \sum_{n=0}^{K-1}\left[\sum_{f=0}^{\infty}\Pr[\tilde{l}=fK+n]z^f\right].$$

We note that the bracketted term in the summation appears in Eq. (29) when $z^{1/K}$ is substituted for z. Hence,

$$\begin{aligned} F(z) &= \sum_{n=0}^{K-1}\left[\frac{1}{K}\sum_{m=0}^{K-1}(z^{1/K}\alpha_m)^{-n}l(z^{1/K}\alpha_m)\right] = \frac{1}{K}\sum_{m=0}^{K-1}\left[\sum_{n=0}^{K-1}(z^{1/K}\alpha_m)^{-n}\right]l(z^{1/K}\alpha_m) \\ &= \frac{1}{K}\sum_{m=0}^{K-1}\frac{1-(z^{1/K}\alpha_m)^{-K}}{1-(z^{1/K}\alpha_m)^{-1}}\,l(z^{1/K}\alpha_m) = \frac{1}{K}\sum_{m=0}^{K-1}\frac{1-z^{-1}}{1-(z^{1/K}\alpha_m)^{-1}}\,l(z^{1/K}\alpha_m) \qquad (33) \\ &= \frac{1-z^{-1}}{K}\sum_{m=0}^{K-1}\frac{l(z^{1/K}\alpha_m)}{1-(z^{1/K}\alpha_m)^{-1}}. \end{aligned}$$

Calculating the expected value of f can be done by taking the derivative of the above equation at $z = 1$ or using the direct approach, that is

$$E[\tilde{f}] = \frac{E[\tilde{l}]-E[\tilde{J}]}{K} = \frac{1}{K}E[\tilde{l}] - \frac{K-1}{2K} + \frac{1}{K}\sum_{m=1}^{K-1}\frac{l(\alpha_m)}{1-\dfrac{1}{\alpha_m}}.$$

APPENDIX B

Starting with Eq. (16) and moving to the Laplace transform domain we obtain

$$D_k^*[s \mid \tilde{w}(k),\tilde{l}(k)] \triangleq E[e^{-s\tilde{D}[k \mid \tilde{w}(k),\tilde{l}(k)]}] = e^{-sT}e^{-s\tilde{w}(k)}e^{-s(T_c/K)\tilde{l}}l^{s(T_c/K)\tilde{J}}e^{-sA(\tilde{J},k)}. \qquad (34)$$

We proceed by eliminating the condition on $\tilde{l}$. Let $l_k[z, \tilde{w}(k)]$ be the generating function of $\tilde{l}$ given $\tilde{w}$. Continuing from Eq. (34) we get

$$\begin{aligned}
D_k^*(s \mid \tilde{w}(k)) &= e^{-sT} e^{-s\tilde{w}(k)} E[e^{-s(T_c/K)^{\tilde{l}}} l^{s(T_c/K)^J} e^{-sA(\tilde{J},k)}] \\
&= e^{-sT} e^{-s\tilde{w}(k)} \left[\sum_{l=0}^{\infty} e^{-s(T_c/K)^{\tilde{l}}} l^{s(T_c/K)^J} e^{-sA(\tilde{J},k)} \Pr[\tilde{l}=l \mid \tilde{w}(k)] \right] \\
&= e^{-sT} e^{-s\tilde{w}(k)} \sum_{f=0}^{\infty} \sum_{j=0}^{K-1} e^{-s(T_c/K)^{(fK+j)}} l^{s(T_c/K)^j} e^{-sA(j,k)} \Pr[\tilde{l}=fK+j \mid \tilde{w}(k)] \\
&= e^{-sT} e^{-s\tilde{w}(k)} \sum_{j=0}^{K-1} e^{-sA(j,k)} \left[\sum_{f=0}^{\infty} (e^{-s(T_c/K)})^{fK} \Pr[\tilde{l}=fK+j \mid \tilde{w}(k)] \right].
\end{aligned}$$

We recognize the bracketted term as the basic relation of Appendix A, with z replaced by $z_s \triangleq e^{-s(T_c/K)}$. Making the substitution we get

$$\begin{aligned}
D_k^*[s \mid \tilde{w}(k)] &= e^{-sT} e^{-s\tilde{w}(k)} \sum_{j=0}^{K-1} e^{-sA(j,k)} \left[\sum_{m=0}^{K-1} \frac{1}{K} (z_s\alpha_m)^{-j} l[z_s\alpha_m, \tilde{w}(k)] \right] \\
&= \frac{1}{K} e^{-sT} e^{-s\tilde{w}(k)} \sum_{m=0}^{K-1} l[z_s\alpha_m, \tilde{w}(k)] \sum_{j=0}^{K-1} e^{-sA(j,k)} (z_s\alpha_m)^{-j} \qquad (35)
\end{aligned}$$

where $\alpha_m = e^{i(2\pi m/K)}$ is the unit root of order K.

The next step is to remove the condition on $\tilde{w}(k)$, but before doing so we must evaluate the generating function of $\tilde{l}$ since it depends on $\tilde{w}(k)$. To do so we notice that given $\tilde{w}(k)$, the total number of packets that are to be transmitted before the tagged packet, $\tilde{l}(k)$, is the sum of three independent random variables: (i) the packets already waiting at the beginning of interval $d(k)$ less one packet (if there were any) that is transmitted in the first slot of $d(k)$, that is, $\tilde{q}(k) - \Delta_{\tilde{q}(k)}$ (generating function $[(1 - z^{-1})Q_k(0) + z^{-1}Q_k(z)]$); (ii) packets arriving from the beginning of interval $d(k)$ until the arrival of the tagged message (generating function $e^{-\lambda[d(k)-\tilde{w}(k)][1-L(z)]}$); (iii) packets of the tagged message, not including the tagged packet itself [generating function $L(z)z^{-1}$]. Therefore,

$$\begin{aligned}
l_k[z, \tilde{w}(k)] &\triangleq E[z^{\tilde{l}(k)} \mid \tilde{w}(k)] \\
&= [(1 - z^{-1})Q_k(0) + z^{-1}Q_k(z)] e^{-\lambda[d(k)-\tilde{w}(k)][1-L(z)]} L(z)z^{-1}. \qquad (36)
\end{aligned}$$

By defining

$$h_k(z) \triangleq [(1 - z^{-1})Q_k(0) + z^{-1}Q_k(z)] e^{-\lambda d(k)[1-L(z)]} L(z)z^{-1} = Q_{k+1}(z)L(z)z^{-1}$$

Eq. (36) can be written as

$$l_k[z, \tilde{w}(k)] = h_k(z) e^{\lambda\tilde{w}(k)[1-L(z)]}$$

and when substituted into (35) we get

$$\begin{aligned}
D_k^*(s \mid \tilde{w}(k)) &= \frac{1}{K} e^{-sT} e^{-s\tilde{w}(k)} \sum_{m=0}^{K-1} h_k(z_s\alpha_m) e^{\lambda\tilde{w}(k)[1-L(z_s\alpha_m)]} \sum_{j=0}^{K-1} e^{-sA(j,k)} (z_s\alpha_m)^{-j} \\
&= \frac{1}{K} e^{-sT} \sum_{m=0}^{K-1} h_k(z_s\alpha_m) \left[\sum_{j=0}^{K-1} e^{-sA(j,k)} (z_s\alpha_m)^{-j} \right] e^{-\tilde{w}(k)\{s-\lambda[1-L(z_s\alpha_m)]\}}.
\end{aligned}$$

(37)

Since $\tilde{w}(k)$ is uniformly distributed between 0 and $d(k)$ we have from (37)

$$
\begin{aligned}
D_k^*(s) &= E[e^{-s\tilde{D}(k)}] = E[D_k^*(s \mid \tilde{w}(k))] = \frac{1}{d(k)} \int_0^{d(k)} D_k^*(s \mid w) dw \\
&= \frac{1}{K} e^{-sT} \sum_{m=0}^{K-1} h_k(z_s\alpha_m) \frac{1}{d(k)} \int_0^{d(k)} e^{-w\{s-\lambda[1-L(z_s\alpha_m)]\}} dw \\
&\quad \times \left[\sum_{j=0}^{K-1} e^{-sA(j,k)} (z_s\alpha_m)^{-j} \right] \\
&= \frac{1}{K} e^{-sT} \sum_{m=0}^{K-1} h_k(z_s\alpha_m) \frac{1 - e^{-d(k)\{s-\lambda[1-L(z_s\alpha_m)]\}}}{d(k)\{s - \lambda[1 - L(z_s\alpha_m)]\}} \left[\sum_{j=0}^{K-1} e^{-sA(j,k)} (z_s\alpha_m)^{-j} \right] \qquad (38) \\
&= \frac{1}{Kd(k)} e^{-sT} \sum_{m=0}^{K-1} Q_{k+1}(z_s\alpha_m) L(z_s\alpha_m) \frac{1 - e^{-d(k)\{s-\lambda[1-L(z_s\alpha_m)]\}}}{z_s\alpha_m\{s - \lambda[1 - L(z_s\alpha_m)]\}} \\
&\quad \times \left[\sum_{j=0}^{K-1} e^{-sA(j,k)} (z_s\alpha_m)^{-j} \right]
\end{aligned}
$$

which is the sought expression.