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TWO COMPETING DISCRETE-TIME QUEUES WITH PRIORITY

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Abstract

This paper considers a class of two discrete-time queues with infinite buffers that compete for a single server. Tasks requiring a deterministic amount of service time, arrive randomly to the queues and have to be served by the server. One of the queues has priority over the other in the sense that it always attempts to get the server, while the other queue attempts only randomly according to a rule that depends on how long the task at the head of the queue has been waiting in that position. The class considered is characterized by the fact that if both queues compete and attempt to get the server simultaneously, then they both fail and the server remains idle for a deterministic amount of time. For this class we derive the steady-state joint generating function of the state probabilities. The queueing system considered exhibits interesting behavior, as we demonstrate by an example.

Keywords: Discrete-time queues, competing queues, priority systems, random access.

1. Introduction

This paper considers a class of two discrete-time queues that compete for a single server. We assume that the two queues have unbounded buffers and that all tasks that arrive to the queues need the same amount of service. Time is divided into equal length intervals, called slots, that correspond to the time needed to serve a single task. Service of a task may begin only at slot boundaries and we assume that in each queue the tasks are served in a first-come-first-served order.

A single server is available for serving the tasks of the two queues. At the beginning of each slot, a nonempty queue decides whether to compete for the server and attempt to get it for the current slot or not. The main feature of our model is that if the two queues decide to compete for the server in a particular slot, then they both fail and the server remains idle during the slot. When only one queue is competing, then it gets the server and the task at the head of this queue is served during that slot and removed from the queue at the end of that slot. Another important feature of the model is that one of the queues (number 2 in this paper) has priority over the other queue, in the sense that it always competes for the server when it is nonempty. The decisions of the other queue

(queue 1) whether to compete for the server or not, are randomized, and to generalize, we assume that they depend on how long the task at the head of its queue has been waiting in that position. Specifically, if the task at the head of queue 1 has been waiting exactly k ($k = 0, 1, 2, \ldots$) slots in that position, then queue 1 competes for the server with probability q_k. We restrict our analysis to systems for which $q_k = p =$ constant for $k \geqslant K$, where K is an arbitrary constant.

We assume that tasks arrive randomly at the two queues. In general, the arrival processes to the two queues may be dependent. Let $A_1(t)$ and $A_2(t)$ be the number of tasks entering queue 1 and queue 2 from their corresponding sources in the time interval $(t, t+1]$, $t = 0, 1, 2, \cdots$. The joint input process $[A_1(t), A_2(t)]$ is assumed to be a sequence of independent and identically distributed random vectors with non-negative integer-valued elements. Let

$$a(i, j) = \operatorname{Prob}(A_1(t) = i, A_2(t) = j) \quad (i \geqslant 0, j \geqslant 0) \tag{1}$$

and

$$F(x, y) = E\left[x^{A_1(t)} y^{A_2(t)}\right] = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a(i, j) x^i y^j \quad (|x| \leqslant 1, |y| \leqslant 1). \tag{2}$$

We assume that $F(x, y)$, the joint generating function of the arrival processes, depends on both x and y, namely that tasks do arrive at the two queues with nonzero probability (otherwise one of the queues will be empty with probability 1) and that the two queues have *infinite buffers*.

It is easy to see that when $N < M$, the case $K = N$ is a special case of $K = M$, since for $0 \leqslant k \leqslant N$ the parameters q_k for $K = M$ can be chosen to be identical to the parameters q_k for $K = N$ and for $N \leqslant k \leqslant M$ we can choose $q_k = q_N$ for $K = M$. Therefore, when K increases, we expect to be able to improve the performance of the system (certainly by properly choosing of the parameters q_k).

The queueing model described above has a variety of applications. The application of specific interest is to the analysis of computer-communication networks and local area networks that use broadcast medium for transmission of information; see Kleinrock [1]. The queues in this case represent the nodes of the network, the tasks are the packets transmitted by the nodes, and the server is the shared broadcast channel through which packets are transmitted. The competition rules correspond to the rules used by the nodes for accessing the shared channel. In that respect, the current model is a substantial generalization of the access method considered by Sidi and Segall [4] that corresponds to $K = 0$.

The current paper is concerned with the description of stochastic aspects of the above class of systems under steady-state conditions. Specifically, we are interested in the state probabilities of the system, the conditions for steady state, the average delays etc. for a broad class of arrival processes. Generating functions are used in order to characterize the state probabilities. As often happens with queues that are coupled (Sidi and Segall [4], Fayolle and Iasnogorodski [7], Eisenberg ([8,9]), some boundary functions need to be determined during the solution process. A substantial effort is devoted in this paper for that task. In addition, we indicate some interesting properties of competing systems with a single server through an accompanying example.

Several discrete-time queueing systems that have been previously investigated are related to our paper. In particular Konheim and Meister [3] and Morrison [2] have considered systems without competitions, and Sidi and Segall [4] have treated a special case of this paper.

2. Steady-state distribution

Fix $K \geqslant 2$. We say that the system is in state 0 if there are no tasks at queue 1 or the task at the head of queue 1 has just arrived to that position. The system is in state k ($1 \leqslant k < K$) if a task has been waiting at the head of queue 1 for exactly k slots, and in state K if it has been waiting there for K slots or more. Let $P_k(m, n)$ ($0 \leqslant k \leqslant K$, $m \geqslant 0$, $n \geqslant 0$) be the equilibrium joint probability that the system is in state k and the queue lengths at queues 1, 2 are m, n respectively at slot boundaries. The equilibrium equations are given by (henceforth $\bar{q}_k = 1 - q_k$):

$$P_0(0, n) = a(0, n)P_0(0, 0) + \sum_{l=0}^{n} a(0, l)P_0(0, n-l+1)$$

$$+a(0, n)\sum_{k=0}^{K} q_k P_k(1, 0) \quad (n \geqslant 1) \tag{3a}$$

$$P_0(m, 0) = a(m, 0)[P_0(0, 0) + P_0(0, 1)]$$

$$+\sum_{k=0}^{K}\sum_{l=0}^{m} q_k a(l, 0)P_k(m-l+1, 0) \quad (m \geqslant 0) \tag{3b}$$

$$P_0(m, n) = a(m, n)P_0(0, 0) + \sum_{l=0}^{n} a(m, l)P_0(0, n-l+1)$$

$$+\sum_{k=0}^{K}\sum_{l=0}^{m} q_k a(l, n)P_k(m-l+1, 0) \quad (m \geqslant 1,\ n \geqslant 1). \tag{3c}$$

For $0 < k < K$ we have

$$P_k(0, n) = 0 \quad (n \geqslant 0) \tag{4a}$$

$$P_k(m, 0) = \sum_{l=0}^{m-1} a(l, 0)[\bar{q}_{k-1}P_{k-1}(m-l, 0) + \bar{q}_{k-1}P_{k-1}(m-l, 1)]$$

$$(m \geqslant 1) \tag{4b}$$

$$P_k(m, n) = \sum_{l=0}^{m-1} \left\{ \sum_{j=0}^{n-1} a(l, j) q_{k-1} P_{k-1}(m-l, n-j) + \sum_{j=0}^{n} a(l, j) \bar{q}_{k-1} P_{k-1}(m-l, n-j+1) \right\} + \sum_{l=0}^{m-1} a(l, n) \bar{q}_{k-1} P_{k-1}(m-l, 0) \quad (m \geqslant 1,\ n \geqslant 1) \tag{4c}$$

and

$$P_K(0, n) = 0 \quad (n \geqslant 0) \tag{5a}$$

$$P_K(m, 0) = \sum_{k=K-1}^{K} \sum_{l=0}^{m-1} a(l, 0)\left[\bar{q}_{k-1} P_{k-1}(m-l, 0) + \bar{q}_{k-1} P_{k-1}(m-l, 1)\right] \quad (m \geqslant 1) \tag{5b}$$

$$P_K(m, n) = \sum_{k=K-1}^{K} \left\{ \sum_{l=0}^{m-1} \left[\sum_{j=0}^{n-1} a(l, j) q_{k-1} P_{k-1}(m-l, n-j) + \sum_{j=0}^{n} a(l, j) \bar{q}_{k-1} P_{k-1}(m-l, n-j+1) \right] + \sum_{l=0}^{m-1} a(l, n) \bar{q}_{k-1} P_{k-1}(m-l, 0) \right\} \quad (m \geqslant 1,\ n \geqslant 1). \tag{5c}$$

REMARK

Since the underlying Markov chain that describes the system is irreducible and aperiodic, the condition for steady-state is that $P_0(0, 0) > 0$.

Let

$$G_k(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P_k(m, n) x^m y^n \quad (0 \leqslant k \leqslant K) \tag{6}$$

be the steady-state queue length joint generating function when the system is in state k. It can be shown (by straightforward manipulation of (3)–(5)) that:

$$G_0(x, y) = F(x, y)\left\{ G_0(0, 0)\left(1 - q_0 x^{-1}\right) + \left[G_0(0, y) - G_0(0, 0)\right] y^{-1} + \sum_{k=0}^{K} q_k G_k(x, 0) x^{-1} \right\} \tag{7a}$$

$$G_1(x, y) = F(x, y)\left\{ \left[G_0(x, y) - G_0(x, 0) - G_0(0, y) + G_0(0, 0)\right] \times \left(q_0 + \bar{q}_0 y^{-1}\right) + \left[G_0(x, 0) - G_0(0, 0)\right] \bar{q}_0 \right\} \tag{7b}$$

$$G_k(x, y) = F(x, y)\left\{ \left[G_{k-1}(x, y) - G_{k-1}(x, 0)\right]\left(q_{k-1} + \bar{q}_{k-1} y^{-1}\right) + G_{k-1}(x, 0) \bar{q}_{k-1} \right\} \quad (2 \leqslant k \leqslant K-1) \tag{7c}$$

and

$$G_K(x, y) = F(x, y)\left\{ \sum_{k=K-1}^{K} \left[[G_k(x, y) - G_k(x, 0)](q_k + \bar{q}_k y^{-1}) + G_k(x, 0)\bar{q}_k \right] \right\}. \tag{7d}$$

In order to uniquely determine $G_k(x, y)$ $(0 \leqslant k \leqslant K)$, we have to find the boundary functions $G_0(0, y)$, $G_k(x, 0)$ for $0 \leqslant k \leqslant K$ (a total of $K+2$ functions) and the constant $G_0(0, 0)$.

Determination of $G_0(0, y)$

Let y_0 designate the solution of the equation $F(0, y)y^{-1} = 1$ in the unit disk $|y_0| < 1$. By Rouche's theorem (Copson [6]) it can be shown that such a solution exists and is unique. We now prove the following:

THEOREM 1

$$G_0(0, y) = F(0, y)\frac{G_0(0, 0)(y_0^{-1} - y^{-1})}{1 - F(0, y)y^{-1}}. \tag{8}$$

Proof

Let $x \to 0$ in (7a). Then

$$G_0(0, y) = F(0, y)\left\{ G_0(0, 0) + [G_0(0, y) - G_0(0, 0)]y^{-1} + \sum_{k=0}^{K} q_k P_k(1, 0) \right\},$$

which gives

$$G_0(0, y) = F(0, y)\frac{G_0(0, 0)(1 - y^{-1}) + \sum_{k=0}^{K} q_k P_k(1, 0)}{1 - F(0, y)y^{-1}}.$$

Since $G_0(0, y)$ is analytic for $|y| < 1$, we have $\sum_{k=0}^{K} q_k P_k(1, 0) = G_0(0, 0)(y_0^{-1} - 1)$, and therefore (8) follows. □

Determination of $G_k(x, 0)$ $(0 \leqslant k \leqslant K)$

The determination of the $K+1$ boundary functions $G_k(x, 0)$ $(0 \leqslant k \leqslant K)$ is much more complex. For convenience let us use the following definitions for $0 \leqslant k \leqslant K$:

$$b_k(y) \triangleq q_k + \bar{q}_k y^{-1}, \quad d_k(y) \triangleq \bar{q}_k - b_k(y). \tag{9}$$

Using these definitions we see from (7d) that:

$$G_K(x, y) = F(x, y)\frac{N(x, y)}{1 - F(x, y)b_K(y)}, \tag{10}$$

with

$$N(x, y) = G_{K-1}(x, y)b_{K-1}(y) + G_K(x, 0)d_K(y) + G_{K-1}(x, 0)d_{K-1}(y).$$

Now recursive substitution of (7c) for $k = K-1,\ K-2, \cdots, 2$ in this last expression yields:

$$N(x, y) = G_K(x, 0)d_K(y) + \sum_{k=1}^{K-1} d_k(y)B_{k+1}(y)F^{K-k-1}(x, y)G_k(x, 0) + B_1(y)F^{K-2}(x, y)G_1(x, y), \tag{11a}$$

where

$$B_k(y) \triangleq \prod_{i=k}^{K-1} b_i(y) \quad 0 \leqslant k \leqslant K-1, \quad B_K(y) \triangleq 1. \tag{11b}$$

Using (7b) in (11a) we obtain

$$N(x, y) = G_K(x, 0)d_K(y) + \sum_{k=0}^{K-1} d_k(y)B_{k+1}(y)F^{K-k-1}(x, y)G_k(x, 0) + F^{K-1}(x, y)\{[G_0(x, y) - G_0(0, y)]B_0(y) - d_0(y)B_1(y)G_0(0, 0)\}. \tag{12}$$

Now using (7a) in (12) we finally obtain:

$$N(x, y) = \sum_{k=0}^{K} D_k(x, y)G_k(x, 0) + M(x, y), \tag{13a}$$

where

$$D_k(x, y) = d_k(y)B_{k+1}(y)F^{K-k-1}(x, y) + B_0(y)F^K(x, y)x^{-1}q_K \quad 0 \leqslant k \leqslant K-1 \tag{13b}$$

$$D_K(x, y) = d_K(y) + B_0(y)F^K(x, y)x^{-1}q_K \tag{13c}$$

$$M(x, y) = F^{K-1}(x, y)\{B_0(y)[y^{-1}F(x, y) - 1]G_0(0, y) + [(1 - q_0x^{-1} - y^{-1})B_0(y)F(x, y) - d_0(y)B_1(y)]G_0(0, 0)\}. \tag{13d}$$

Note that the functions $D_k(x, y)$ $(0 \leqslant k \leqslant K)$ are all known and $M(x, y)$ is known up to the constant $G_0(0, 0)$.

We now state a theorem whose proof appears in the appendix.

THEOREM 2

Each boundary function $G_k(x, 0)$ $(1 \leqslant k \leqslant K)$ can be expressed as a linear combination of $G_0(x, 0)$ as

$$G_k(x, 0) = C_k(x) G_0(x, 0) + H_k(x) \quad (1 \leqslant k \leqslant K), \tag{14}$$

where the functions $C_k(x)$ and $H_k(x)$ for $1 \leqslant k \leqslant K$ are known. □

Define $C_0(x) \triangleq 1$ and $H_0(x) \triangleq 0$. Substitution of (14) in (13a) yields:

$$N(x, y) = G_0(x, 0) \sum_{k=0}^{K} D_k(x, y) C_k(x) + \tilde{M}(x, y), \tag{15a}$$

where

$$\tilde{M}(x, y) = M(x, y) + \sum_{k=0}^{K} D_k(x, y) H_k(x). \tag{15b}$$

Now for $|x| < 1$ let $f = f(x)$ be the unique solution of $F(x, f) b_K(f) = 1$ with the property $|f| < 1$. It can be shown by Rouche's theorem (Copson [6]) that such a solution exists and is unique. Since $G_K(x, y)$ is analytic in the polydisk $|x| < 1$, $|y| < 1$, it is clear from (10) that $N(x, f) = 0$ so that from (15a) we have

THEOREM 3

$$G_0(x, 0) = -\frac{\tilde{M}(x, f)}{\sum_{k=0}^{K} D_k(x, f) C_k(x)}. \quad \square \tag{16}$$

Thus $G_0(x, 0)$ is determined up to the constant $G_0(0, 0)$. Consequently, from theorem 2, it is clear that $G_k(x, 0)$ $1 \leqslant k \leqslant K$ are all determined up to the constant $G_0(0, 0)$. Finally, $G_0(0, 0)$ is determined by using the normalization condition

$$\sum_{k=0}^{K} G_k(1, 1) = 1. \tag{17}$$

Now that we have determined the boundary functions $G_0(0, y)$, $G_k(x, 0)$ $(0 \leqslant k \leqslant K)$ and the constant $G_0(0, 0)$ we see from (7) that $G_k(x, y)$ $(0 \leqslant k \leqslant K)$ are all uniquely determined. The steady-state generating function of the queue lengths at the queues at a random slot boundary is

$$G(x, y) = \sum_{k=0}^{K} G_k(x, y).$$

From $G(x, y)$ any moment of the queue lengths at the queues can, in principle, be derived.

REMARK

Although the analysis in this section was restricted to $K \geqslant 2$, it is easy to see that the cases $K=0, 1$ were implicitly included in the analysis, simply by choosing $q_0 = q_1 = q_2$ and $q_1 = q_2$, respectively.

3. Independent Bernoulli arrival processes

We now illustrate the results of the previous section by a particular example where the arrivals into queues 1 and 2 are independent Bernoulli trials with probability of an arrival equal to r_1 and r_2 respectively, so that

$$F(x, y) = (xr_1 + \bar{r}_1)(yr_2 + \bar{r}_2) \quad (\bar{r} = 1 - r). \tag{18}$$

For this case we immediately find from (8) that

$$G_0(0, y) = G_0(0, 0)[1 + yr_2/\bar{r}_2]. \tag{19}$$

Using the results of the appendix we find that

$$G_k(x, 0) = F_1^k(x)[G_0(x, 0) - G_0(0, 0)]\alpha_k \ (1 \leqslant k \leqslant K-1) \tag{20a}$$

and

$$G_K(x, 0) = \left\{[G_0(x, 0) - G_0(0, 0)]\left[1 - F_1(x)\bar{r}_2 x^{-1}\left(q_0 + \sum_{k=1}^{K-1} q_k F_1^k(x)\alpha_k\right)\right] + G_0(0, 0)[1 - F_1(x)]\right\} / F_1(x)\bar{r}_2\bar{q}_K x^{-1}, \tag{20b}$$

where $F_1(x) = xr_1 + \bar{r}_1$ and the coefficients α_k are given by:

$$\alpha_k = \alpha_{0,k} \quad (1 \leqslant k \leqslant K-1), \tag{20c}$$

with the coefficients $\alpha_{0,k}$ determined via the following recursion:

$$\alpha_{n,0} = \begin{cases} 1 & n=0 \\ r_2/\bar{r}_2 & n=1 \\ 0 & 2 \leqslant n \leqslant K-1 \end{cases} \tag{20d}$$

$$\alpha_{n,k} = \frac{1}{n+1}\bar{q}_{k-1}\sum_{i=1}^{n+1}\binom{n+1}{i}F_2^{(n+1-i)}(0)\alpha_{i,k-1} + q_{k-1}\sum_{i=1}^{n}\binom{n}{i}F_2^{(n-i)}(0)\alpha_{i,k-1} + \bar{q}_{k-1}F_2^{(n)}(0)\alpha_{0,k-1} \tag{20e}$$

$$(0 \leqslant n \leqslant K^{-k-1}, 1 \leqslant k \leqslant K-1)$$

and $F_2^{(0)}(0) = \bar{r}_2$, $F_2^{(1)}(0) = r_2$ and $F_2^{(n)}(0) = 0 \quad n \geqslant 2$.

Using (16), $G_0(x, 0)$ is determined up to the constant $G_0(0, 0)$. After tedious algebra, a closed form expression for the constant $G_0(0, 0)$ is found to be:

$$G_0(0, 0) = \bar{r}_2 \left\{ 1 - r_1 \frac{\beta_K + \bar{r}_2 \sum_{k=0}^{K-1} \alpha_k (\beta_k q_K - \beta_K q_k)}{q_K(\bar{q}_K - r_2)} \right\}, \tag{21a}$$

where the constants β_k $(0 \leqslant k \leqslant K)$ are given by:

$$\beta_k = (k+1) q_k \bar{q}_K + \bar{q}_k - q_k \sum_{j=0}^{k} \bar{q}_j \quad 0 \leqslant k \leqslant K-1,$$

$$\beta_k = K q_K \bar{q}_K + \bar{q}_K - q_K \sum_{j=0}^{K-1} \bar{q}_j. \tag{21b}$$

Specifically, if we define $q_K = p$ we obtain:

$$G_0(0, 0) = \bar{r}_2 \left[1 - \frac{r_1 \bar{p}}{p(\bar{p} - r_2)} \right] \quad \text{for } K = 0 \tag{22}$$

$$G_0(0, 0) = \bar{r}_2 \left[1 - r_1 \frac{\bar{p} + (p - q_0)(\bar{p} - r_2)}{p(\bar{p} - r_2)} \right] \quad \text{for } K = 1 \tag{23}$$

$$G_0(0, 0) = \bar{r}_2 \left[1 - r_1 \frac{2p\bar{p} + \bar{p} - p(\bar{q}_0 + \bar{q}_1) + \bar{r}_2(p - q_0 + (p - q_1)(1 - q_0\bar{p}))}{p(\bar{p} - r_2)} \right]$$

for $K = 2$. (24)

Recall that for $K = 0$, queue 1 competes and attempts to get the server, when nonempty, with constant probability p. For $K = 1$, queue 1 competes for the server when a task just arrives to the head of its queue with probability q_0 and if it doesn't get it, it continues to attempt to get it with probability p. For $K = 2$, queue 1 competes for the server when a task just arrives to the head of the queue with probability q_0, if it doesn't get it, it attempts again with probability q_1 and afterwards with constant probability p.

From (23) we see that for $K = 1$, $G_0(0, 0)$ is a monotonic increasing function of q_0 for all r_1, r_2 and p. Therefore, when $K = 1$ $G_0(0, 0)$ is maximized for $q_0 = 1$. For $K = 2$ we have found from (24) (by numerical search) that $q_0 = 1$, $q_1 = 0$ maximize $G_0(0, 0)$ for all values of r_1, r_2 and p.

The explicit expressions for the average time delays in the system (that are derived from the average queue lengths at the queues by Little's [5] result) are too complicated to be given here. However we shall present several numerical results for $K = 0, 1, 2$. A somewhat surprising result for $K = 1$ is that the average time delay at queue 2 is independent of q_0. The average time delay at queue 1 and

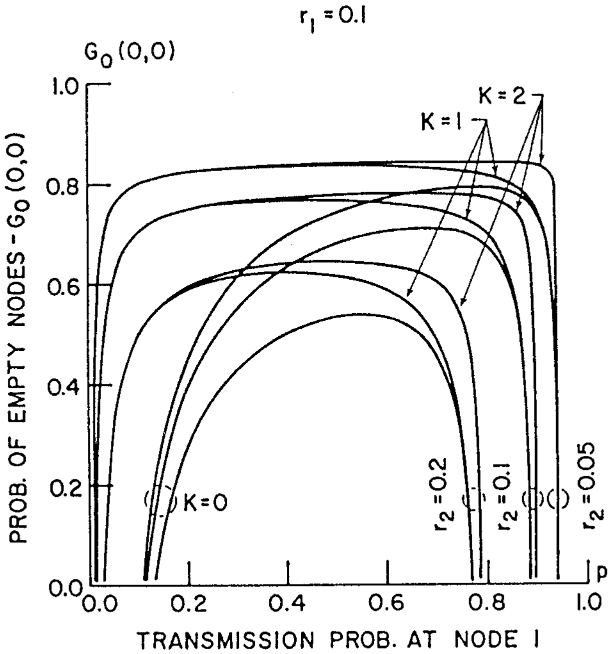

Fig. 1. $K = 0, 1, 2$: $G_0(0, 0)$ vs. p $(q_0 = 1,\ q_1 = 0)$.

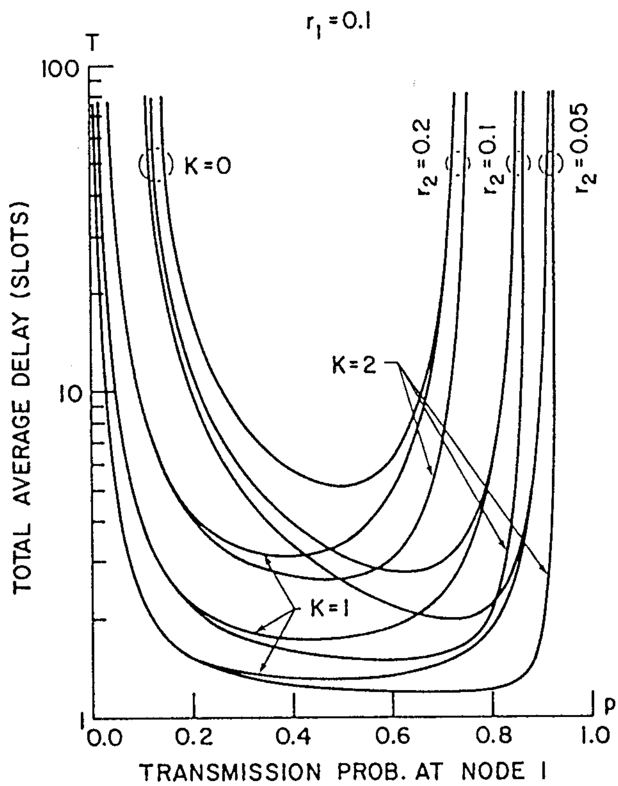

Fig. 2. $K = 0, 1, 2$: T vs. p $(q_0 = 1,\ q_1 = 0)$.

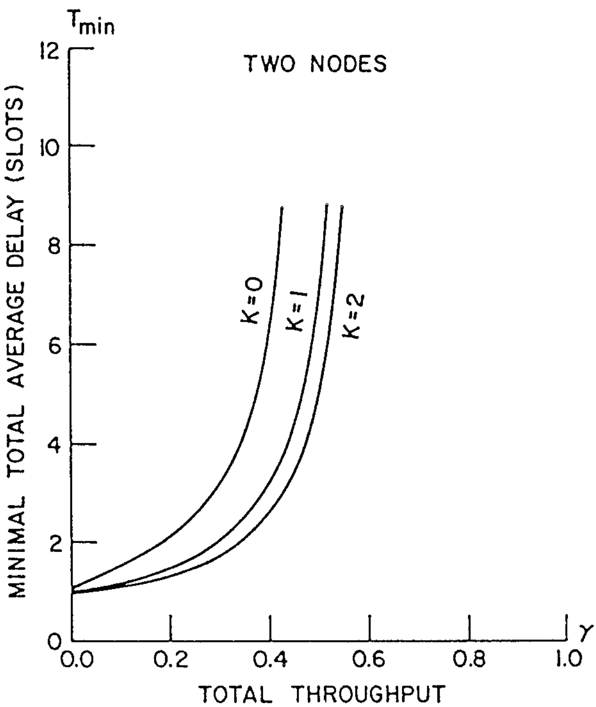

Fig. 3. $K = 0, 1, 2$: T_{min} vs. γ.

hence the total average delay in the system are decreasing functions of q_0. Consequently, to minimize the total average delay for $K = 1$ we have to choose $q_0 = 1$. For $K = 2$ we found that the total average delay is minimized when $q_0 = 1$, $q_1 = 0$. The fact that the optimal q_1 is $q_1 = 0$ is not surprising, since if queue 1 attempts to get the server with probability 1 ($q_0 = 1$) then if it is unsuccessful, it indicates that queue 2 is nonempty. Therefore, in order to insure that queue 2 will get the server, queue 1 must forgive ($q_1 = 0$) for one slot.

Comparisons between the three cases appear in Figs. 1–3. In all these figures we choose $q_0 = 1$ when $K = 1$ and $q_0 = 1$, $q_1 = 0$ when $K = 2$. In Fig. 1 the probability for empty system $G_0(0, 0)$ is plotted versus p for different arrival rates. In fig. 2 the total average time delay T is plotted versus p. From these figures we see that as K increases the performance of the system is improved. We also note that both T and $G_0(0, 0)$ are almost independent of p for low arrival rates (except for very large or very small values of p). The reason is that in this case most of the tasks at queue 1 are served as soon as they arrive to the head of the queue. However, in each of the three cases the parameter p plays a role. We can find the optimal p that minimizes the total average delay for any arrival rates. In fig. 3 the *minimal* total average delay is plotted versus the total arrival rate - γ when $r_1 = r_2 = r$ (clearly $\gamma = 2r$) for the three cases ($K = 0, 1, 2$). As expected we see again that the performance of the system improves when K

increases. Finally, we note that when $r_1 = r_2 = r$, the steady-state conditions for $K = 0, 1, 2$ are that $r < 0.25$, $r < 0.295$, and $r < 0.309$, respectively.

4. Extensions

Some extensions of the system of the two competing queues that was investigated in this paper are of interest. For instance, one extension that can be easily analyzed is to incorporate routing into the system by allowing a serviced packet at some queue to join the other queue with some probability or to leave the system with some probability. Another extension, that does not seem to be easy to analyze, is to allow arbitrary number of slots (not necessarily constant) needed to serve packets rather than a single slot (the assumption made in this paper). An assumption of geometric service time might be reasonable as a first step in this direction. The most challenging extension is to allow both queues of the system to compete randomly for the server, namely that none of queues will have priority over the other. A first step towards this direction is given by Nain [10], where it is assumed that both queues compete for the server with some constant probability and that the arrivals are independent Bernoulli trials.

5. Appendix

In this appendix we prove theorem 2, namely, that each of the boundary functions $G_k(x, 0)$ $(1 \leqslant k \leqslant K)$ can be expressed as a linear combination of $G_0(x, 0)$ and a known function. We use the notation $G_k^{(n)}(x, y)$ and $F^{(n)}(x, y)$ to denote the n-th derivative of $G_k(x, y)$ and $F(x, y)$, respectively, with respect to y. From (7a) we see that:

$$yG_0(x, y) = F(x, y)\left[yg(x) + G_0(0, y) - G_0(0, 0)\right], \tag{A.1}$$

where

$$g(x) = G_0(0, 0)\left(1 - q_0 x^{-1}\right) + x^{-1} \sum_{k=0}^{K} q_k G_k(x, 0). \tag{A.2}$$

Therefore:

$$\begin{aligned} &yG_0^{(n)}(x, y) + nG_0^{(n-1)}(x, y) \\ &\quad = F^{(n)}(x, y)\left[yg(x) + G_0(0, y) - G_0(0, 0)\right] \\ &\qquad + nF^{(n-1)}(x, y)\left[g(x) + G_0^{(1)}(0, y)\right] \\ &\qquad + \sum_{i=2}^{n} \binom{n}{i} F^{(n-i)}(x, y) G_0^{(i)}(0, y). \end{aligned} \tag{A.3}$$

Let $y \to 0$ in (A.3). Then

$$G_0(x, 0) = F(x, 0)[g(x) + P_0(0, 1)] \tag{A.4}$$

and for $1 \leqslant n \leqslant K-1$:

$$G_0^{(n)}(x, 0) = G_0(x, 0) c_{n,0}(x) + h_{n,0}(x), \tag{A.5}$$

where

$$c_{n,0}(x) = F^{(n)}(x, 0)/F(x, 0) \tag{A.6}$$

$$h_{n,0}(x) = \frac{1}{n+1} \sum_{i=2}^{n+1} \binom{n+1}{i} i! F^{(n+1-i)}(x, 0) P_0(0, i). \tag{A.7}$$

Recall that $P_k(m, n)$ are the equilibrium probabilities. Now from (7b) we see that

$$\begin{aligned} yG_1(x, y) = F(x, y)\{&[G_0(x, y) - G_0(x, 0) - G_0(0, y) \\ &+ G_0(0, 0)](q_0 y + \bar{q}_0) + y\bar{q}_0[G_0(x, 0) - G_0(0, 0)]\}. \end{aligned} \tag{A.8}$$

Therefore:

$$\begin{aligned} &yG_1^{(n)}(x, y) + nG_1^{(n-1)}(x, y) \\ &\quad = (q_0 y + \bar{q}_0) \sum_{i=0}^{n} \binom{n}{i} F^{(n-i)}(x, y)[G_0^{(i)}(x, y) - G_0^{(i)}(0, y)] \\ &\qquad + nq_0 \sum_{i=1}^{n-1} \binom{n-1}{i} F^{n-1-i}(x, y)[G_0^{(i)}(x, y) - G_0^{(i)}(0, y)] \\ &\qquad + \bar{q}_0[G_0(x, 0) - G_0(0, 0)][yF^{(n)}(x, y) + nF^{(n-1)}(x, y)]. \end{aligned} \tag{A.9}$$

Let $y \to 0$ in (A.9). Then

$$\begin{aligned} G_1^{(n-1)}(x, 0) = &\frac{1}{n}\bar{q}_0 \sum_{i=1}^{n} \binom{n}{i} F^{(n-i)}(x, 0)[G_0^{(i)}(x, 0) - i!P_0(0, i)] \\ &+ q_0 \sum_{i=1}^{n-1} \binom{n-1}{i} F^{(n-1-i)}(x, 0)[G_0^{(i)}(x, 0) - i!P_0(0, i)] \\ &+ \bar{q}_0[G_0(x, 0) - G_0(0, 0)] F^{(n-1)}(x, 0). \end{aligned} \tag{A.10}$$

From (A.10) we finally obtain for $0 \leqslant n \leqslant K-2$ that

$$G_1^{(n)}(x, 0) = c_{n,1}(x) G_0(x, 0) + h_{n,1}(x), \tag{A.11}$$

where

$$\begin{aligned} c_{n,1}(x) = \bar{q}_0 F^{(n)}(x, 0) + &\frac{1}{n+1}\bar{q}_0 \sum_{i=1}^{n+1} \binom{n+1}{i} F^{(n+1-i)}(x, 0) c_{i,0}(x) \\ &+ q_0 \sum_{i=1}^{n} \binom{n}{i} F^{(n-i)}(x, 0) c_{i,0}(x) \end{aligned} \tag{A.12}$$

$$h_{n,1}(x) = \frac{1}{n+1}\bar{q}_0 \sum_{i=1}^{n+1} \binom{n+1}{i} F^{(n+1-i)}(x, 0)[h_{i,0}(x) - i!P_0(0, i)]$$
$$+ q_0 \sum_{i=1}^{n} \binom{n}{i} F^{(n-i)}(x, 0)[h_{i,0}(x) - i!P_0(0, i)]$$
$$- \bar{q}_0 G_0(0, 0) F^{(n)}(x, 0). \qquad \text{(A.13)}$$

Now from (7c) in the same way we obtained (A.10) we have for $2 \leqslant k \leqslant K-1$:

$$G_k^{(n-1)}(x, 0) = \frac{1}{n}\bar{q}_{k-1} \sum_{i=1}^{n} \binom{n}{i} F^{(n-i)}(x, 0) G_{k-1}^{(i)}(x, 0)$$
$$+ q_{k-1} \sum_{i=1}^{n-1} \binom{n-1}{i} F^{(n-1-i)}(x, 0) G_{k-1}^{(i)}(x, 0)$$
$$+ \bar{q}_{k-1} G_{k-1}(x, 0) F^{(n-1)}(x, 0). \qquad \text{(A.14)}$$

Assume that

$$G_{k-1}^{(n)}(x, 0) = c_{n,k-1}(x) G_0(x, 0) + h_{n,k-1}(x) \quad (0 \leqslant n \leqslant K-k). \qquad \text{(A.15)}$$

We have proved (A.15) for $k = 1, 2$ in (A.5) and (A.11). Substituting (A.15) in (A.14) yields:

$$G_k^{(n)}(x, 0) = c_{n,k}(x) G_0(x, 0) + h_{n,k}(x) \quad (0 \leqslant n \leqslant K-k-1), \qquad \text{(A.16)}$$

where

$$c_{n,k}(x) = \frac{1}{n+1}\bar{q}_{k-1} \sum_{i=1}^{n+1} \binom{n+1}{i} F^{(n+1-i)}(x, 0) c_{i,k-1}(x)$$
$$+ q_{k-1} \sum_{i=1}^{n} \binom{n}{i} F^{(n-i)}(x, 0) c_{i,k-1}(x) + \bar{q}_{k-1} c_{0,k-1}(x) F^{(n)}(x, 0)$$
$$\text{(A.17)}$$

$$h_{n,k}(x) = \frac{1}{n+1}\bar{q}_{k+1} \sum_{i=1}^{n+1} \binom{n+1}{i} F^{(n+1-i)}(x, 0) h_{i,k-1}(x)$$
$$+ q_{k+1} \sum_{i=1}^{n} \binom{n}{i} F^{(n-i)}(x, 0) h_{i,k-1}(x) + \bar{q}_{k-1} h_{0,k-1}(x) F^{(n)}(x, 0).$$
$$\text{(A.18)}$$

Then by induction (A.15) is proved for $0 \leqslant k \leqslant K-1$. So we have shown that for $0 \leqslant k \leqslant K-1$ we have:

$$G_k(x, 0) = C_k(x) G_0(x, 0) + H_k(x), \qquad \text{(A.19)}$$

where

$$C_0(x) = 1; \quad H_0(x) = 0 \qquad \text{(A.20)}$$
$$C_k(x) = c_{0,k}(x); \; H_k(x) = h_{0,k}(x) \quad (1 \leqslant k \leqslant K-1).$$

Finally, from (7a) using (A.19) we obtain:

$$G_K(x, 0) = C_K(x) G_0(x, 0) + H_K(x), \tag{A.21}$$

where

$$C_K(x) = \left[x - F(x, 0) \sum_{k=0}^{K-1} q_k C_k(x) \right] / q_K F(x, 0) \tag{A.22}$$

$$H_K(x) = -F(x, 0) \left[x P_0(0, 1) + G_0(0, 0)(x - q_0) + \sum_{k=0}^{K-1} q_k H_k(x) \right] / q_K F(x, 0), \tag{A.23}$$

thus proving theorem 2. □

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