# The Canonical Approximation in the Performance Analysis of Packet Radio Networks

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## ABSTRACT

The purpose of this paper is to present the applications of the canonical approximation technique to performance analysis of multihop packet radio networks. The canonical approximation gives a *closed − form* approximation of global performance measures. The computational complexity of the method is *independent* of the size of the network, whereas the precision increases exponentially with the size of the system. The method is applied to analyze some packet radio networks operating under CSMA with perfect capture and C-BTMA protocols.

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# 1. INTRODUCTION

In this paper we apply the method of canonical approximation to the performance analysis of multihop packet radio networks. The method is used to analyze some packet radio networks operating under CSMA with perfect capture and C-BTMA protocols.

A packet radio network (PRNET) consists of geographically distributed radio units broadcasting data packets over a limited range. A key design problem of a PRNET is to resolve interference which occurs whenever two or more nodes try to transmit over the shared channel within the same neighborhood. This is accomplished by means of a multiple access protocol - a set of rules which define the process by which a node proceeds to transmit.

Given an access protocol and a packet radio network, one would like to compute a number of important performance measures. The primary performance measure is the throughput - the average number of packets delivered successfully per unit time. Other measures of interest include steady-state probability distribution of the number of transmissions, the fraction of the channel capacity used for successful transmission, the probability that a scheduled transmission is successful and average number of packets in the system.

Most of the work on multiple access protocols has been confined to the single hop case: a transmission of each node may interfere with transmissions of all other nodes. The work on the performance analysis of protocols in multihop environment, where spatial reuse of the shared channel is possible, is still in progress ([Boor80, Braz85, Toba80, Toba83, Silv83]). In practice, different schemes can be analyzed only numerically or through simulation and only for very small and simple networks. In this paper we present a new method to approximate performance measures of interest. The computational complexity of the method is independent of the size of the network, whereas the precision increases exponentially with the size of the system. The method is applied to analyze some packet radio networks operating under CSMA with perfect capture and C-BTMA protocols.

## 2. THE MODEL

The model of multihop packet radio networks that is used here is the one introduced by Boorstyn and Kerschenbaum ([Boor80]). In this model the network consists of $N$ nodes† with a specified "hearing matrix". For any two nodes $i$ and $j$ the hearing matrix specifies whether or not $i$ can hear $j$. The points in time when new and retransmitted packets are scheduled for transmission are called scheduling points. Packets are retransmitted either because at some scheduling point they were inhibited from transmission or because their transmission has been interfered. The process of scheduling points from a node is assumed to be Poisson with parameter $\lambda$. In general, one need not to assume the same rate of $\lambda$ for all the nodes. The lengths of packets are assumed to be distributed exponentially with parameter $\mu$. For notational convenience, we assume $\mu = 1$. The model assumes negligible propagation delay.

Two protocols are considered - Carrier Sense Multiple Access (CSMA) with perfect capture and Conservative Busy Tone Multiple Access (C-BTMA).

Under CSMA, a node wishing to transmit senses the channel. If the channel is sensed idle, the node starts transmitting, else it waits for the next scheduled point in time and repeats the above procedure. Under the perfect capture assumption†, the transmission of a packet from $i$ to $j$ may not be successful only if any of the "hidden nodes" $k$ (neighbors of $j$ but not of $i$) are transmitting to $j$ at the time $i$ starts its transmission.

Under the C-BTMA, any node that senses carrier emits a busy tone. If a node $i$ transmits a packet to node $j$, all the other neighbors of $i$ transmit busy tone, thus blocking all nodes in a region within twice the "hearing radius" of node $i$. Note that under C-BTMA, once a node starts transmitting, it is guaranteed of success.

For these protocols with the above assumptions of Poisson arrivals and exponential service time, it can be shown ([Boor80, Toba83]) that the equilibrium probability distribution $\pi(i)$ of having $i$ simultaneous transmissions in the system is given by

†$N$ will be used to indicate both the set of nodes and network size as long as no confusion arises.

†The capture assumption is defined as the ability of the receiver to correctly receive a packet despite the presence of other time overlapping transmissions. Perfect capture is the ability of receiving correctly the first packet regardless of future overlapping packets, whereas zero capture means the complete destruction of the first packet by any overlapping transmission. We will consider only perfect capture in this paper.

$$\pi(i) = \frac{\rho^i}{Z_N} \tag{1}$$

where $\rho = \lambda/\mu$ and $Z_N$, the "partition" function of the system is given by:

$$Z_N = \sum_{i=0}^{N} \alpha_N^i \rho^i \tag{2}$$

where $\alpha_N^i$ denotes the number of ways to have $i$ concurrent active nodes. The partition function is a generating function for the concurrency levels of the system.

A number of important measures can be obtained once the partition function is computed. The most important performance measure in a packet radio network is the nodal throughput $S_i$ which is defined as the average number of successful transmissions processed by node $i$ per unit time. Note that it is not just the average number of concurrent transmissions, since some of these will not be received by their destinations. The maximum node throughput is called the node capacity.

To calculate the node throughput, let us first calculate the link throughput $S_{i,j}$, the average number of successful transmissions over $i$-to-$j$ link per unit time. Let $A_{i,j}$ denotes the set of nodes that must be silent at the initiation of the $i$-to-$j$ transmission. The probability of success of $i$-to-$j$ transmission is then

$$P_{i,j} = P(A_{i,j} \text{ idle}) = \frac{\sum_{S \subseteq N \setminus A_{i,j}} \rho^{|S|}}{Z_N} = \frac{Z_{N \setminus A_{i,j}}}{Z_N} \tag{3}$$

Note that if the set of nodes $N$ can be represented as $N = N_1 \cup N_2$ where $N_1$ and $N_2$ do not interfere with each other, then

$$Z_N = Z_{N_1} Z_{N_2} \tag{4}$$

If $\rho_{ij}$ denotes the traffic intensity for the packets from node $i$ to node $j$, then from (1) one obtains

$$S_{i,j} = \rho_{ij} P_{i,j} = \rho_{ij} \frac{Z_{N \setminus A_{i,j}}}{Z_N} \tag{5}$$

And therefore, the nodal throughput is given by

$$S_i = \sum_j S_{i,j} \tag{6}$$

## 3. CANONICAL APPROXIMATION

From the above discussion it is clear that the main difficulty in analyzing networks operating under CSMA with perfect capture and C-BTMA is the computation of the partition function. Usually, the partition function is difficult to express in a closed form. To overcome this problem, we propose the method of **Canonical Approximation.** The term canonical is borrowed from statistical physics where a similar method is used to show the equivalence of canonical and grand canonical ensembles ([Path84]).

To apply the method, one first computes the generating function of $Z_N$,

$$Z_G(t) = \sum_{N=0}^{\infty} Z_N t^N \tag{7}$$

By analogy with physics, one calls $Z_G(t)$ the grand partition function. This function is usually much easier to compute than $Z_N$. By Cauchy's theorem

$$Z_N = \frac{1}{2\pi i} \oint \frac{Z_G(t)}{t^{N+1}} dt \tag{8}$$

Assuming that $Z_G(t)$ is a meromorphic function whose smallest (in magnitude) pole $t_0$ is real, positive and of order 1, one can approximate the partition function as follows ([Henr77]):

$$Z_N \approx \frac{-Res[Z_G(t_0)]}{t_0^{N+1}} \tag{9}$$

where $Res[Z_G(t_0)]$ denotes the residue of $Z_G(t)$ at $t_0$. In Appendix I we prove the correctness of the above approximation and show that its (relative) error is decreasing exponentially when $N$ increases.

# 4. APPLICATIONS

## 4.1. TANDEM NETWORKS: CSMA

Consider a tandem of $N$ packet radios operating under the Carrier Sense Multiple Access (CSMA) scheme ([Boor 80]). In such a system all nodes share the same bandwidth and each node (except for the end nodes) can communicate with two neighbors. To calculate the partition function $Z_N$ one applies the canonical approximation as follows.

Step 1. To calculate the grand partition function, derive a recursive relation for $\alpha_N^i$, the number of configurations involving $i$ transmissions. To that end, suppose one more node is added to a tandem of size $N$. Let us examine a configuration involving $i$ transmissions. There are clearly two cases to consider:

Case 1: the $(N+1)$-st radio is not transmitting. There are $\alpha_N^i$ such configurations.

Case 2: the $(N+1)$-st radio is involved in a transmission. There are $\alpha_{N-1}^{i-1}$ such configurations.

Therefore

$$\begin{aligned} &\alpha_{N+1}^i = \alpha_{N-1}^{i-1} + \alpha_N^i \qquad \text{for } N \geq 1, 1 \leq i \leq N \\ &\alpha_0^0 = \alpha_N^0 = 1 \end{aligned} \tag{10}$$

This implies the following recursive relation

$$Z_{N+1} = Z_N + \rho Z_{N-1}, \qquad Z_0 = 1, Z_1 = 1 + \rho \tag{11}$$

Therefore, the grand partition function is

$$Z_G(t) = \sum_{N=0}^{\infty} Z_N t^N = \frac{1 + t\rho}{1 - t - \rho t^2} \tag{12}$$

Step 2. Find the smallest positive pole of the grand partition function.

$$t_0 = \frac{2}{1 + \sqrt{1 + 4\rho}} \tag{13}$$

Step 3. Compute the residue of the grand partition function at $t_0$

$$Res[Z_G(t_0)] = -\frac{1+\rho t_0}{2\rho t_0 + 1} = -\frac{1+\sqrt{1+4\rho}}{2\sqrt{1+4\rho}} \tag{14}$$

Step 4. The partition function is given by

$$Z_N \approx -\frac{Res[Z_G(t_0)]}{t_0^{N+1}} = \frac{1}{\sqrt{1+4\rho}}\left(\frac{1+\sqrt{1+4\rho}}{2}\right)^{N+2} \tag{15}$$

For this particular example, one can solve the simple recurrence equation (11) to get an exact expression for the partition function

$$Z_N = \frac{1}{\sqrt{1+4\rho}}\left[\left(\frac{1+\sqrt{1+4\rho}}{2}\right)^{N+2} - \left(\frac{1-\sqrt{1+4\rho}}{2}\right)^{N+2}\right]$$

The relative error of the approximation is therefore

$$\text{Error} = \frac{1}{\sqrt{1+4\rho}}\left(\frac{\sqrt{1+4\rho}-1}{\sqrt{1+4\rho}+1}\right)^{N+2} \mapsto 0 \tag{16}$$

and it decreases exponentially with $N$.

To calculate the link throughput $S_{i,i+1}$ under the perfect capture assumption, consider a typical node $i$. The transmission of $i$ to node $i+1$ will be successful, if at the start of the transmission, all of the nodes in $A_{i,j} = \{i-1, i, i+1, i+2\}$ are silent. The sets of activities generated by the subsets of nodes $N_1 = \{1, \ldots, i-2\}$ and $N_2 = \{i+3, \ldots, N\}$ are mutually non-interfering and correspond to two tandems of sizes $i-2$ and $N-i-2$ respectively. Therefore, using equations (3) and (7) one finds the probability of successful transmission from node $i$ to node $i+1$ is

$$P_{i,i+1} = \frac{Z_{i-2}Z_{N-i-2}}{Z_N} \approx \frac{1}{\sqrt{1+4\rho}}\left(\frac{2}{1+\sqrt{1+4\rho}}\right)^2 \tag{17}$$

Assuming that node $i$ is equally likely to transmit to node $i+1$ as to node $i-1$, (that is $\rho_{i,i+1} = \rho/2$) the link throughput is given by

$$S_{i,i+1} = \frac{\rho}{2}P_{i,i+1} \approx \frac{\rho}{2\sqrt{1+4\rho}}\left(\frac{2}{1+\sqrt{1+4\rho}}\right)^2 \tag{18}$$

The nodal throughput is obviously $S_i = 2S_{i,i+1}$. Figure 1 gives the graph for the throughput as a function of the load. The link capacity (maximal link throughput) of $S_{i,i+1} = 0.0857$ is achieved at $\rho = 1.2$. For $\rho = 1$ the throughput is 0.085 as has been shown in [Boor80, Toba83] by using numerical methods and simulation. However, one gains a slight improvement in the throughput if packet retransmission attempts are generated faster than the average transmission duration time ($\rho = 1.2$).

## 4.2. TANDEM NETWORKS: C-BTMA

Let's consider the tandem of $N$ packet radios as before, but assume that it operates under the C-BTMA protocol. This means that a node can transmit only if its immediate neighbors as well as the neighbors of the immediate neighbors are silent. As before, one can derive a recurrence relation for the partition function by adding one more node to the tandem and examining the number of configurations having $i$ transmissions. One would get

$$\begin{aligned} &\alpha_{N+1}^{i} = \alpha_N^{i} + \alpha_{N-2}^{i-1} \qquad N \geq 2 \\ &\alpha_0^0 = \alpha_1^0 = \alpha_N^0 = 1 \end{aligned} \tag{19}$$

The partition function then satisfies

$$\begin{aligned} &Z_{N+1} = Z_N + \rho Z_{N-2} \qquad N \geq 2 \\ &Z_0 = 1, Z_1 = 1 + \rho, Z_2 = 1 + 2\rho \end{aligned} \tag{20}$$

From the above, the grand partition function satisfies:

$$Z_G(t) = \frac{1 + \rho t + \rho t^2}{1 - t - \rho t^3} \tag{21}$$

Let $t_0$ be the smallest (positive) pole of $Z_G(t)$ (See Appendix 1 for the existence and uniqueness of this pole). In Appendix 1 we show that the residue of $Z_G(t)$ at $t_0$ is given by

$$Res[Z_G(t_0)] = -\frac{1 + \rho t_0 + \rho t_0^2}{3\rho t_0^2 + 1} \tag{22}$$

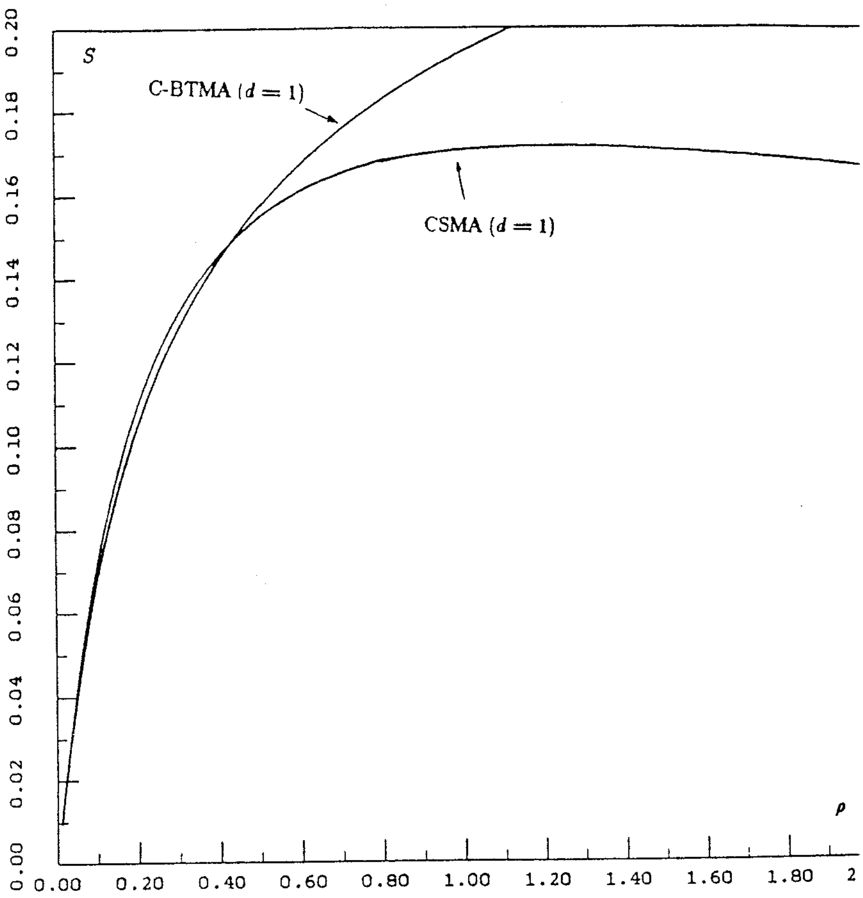


**Figure 1:** Nodal Throughput: CSMA and C-BTMA (d=1)

Applying the canonical approximation, one gets the following expression for the partition function

$$Z_N \approx -\frac{Res[Z_G(t_0)]}{t_0^{N+1}} = \frac{1+\rho t_0+\rho t_0^2}{(3\rho t_0^2+1)t_0^{N+1}} \tag{23}$$

To calculate the throughput, consider a typical node $i$ in a tandem. This node will be successful in a transmission to node $i+1$ if the set of nodes one and two hops away $A_{i,i+1} = \{i-2, i-1, i, i+1, i+2\}$ are silent. The sets of activities generated by the subsets of nodes $N_1 = \{1, 2, \ldots, i-3\}$ and $N_2 = \{i+3, \ldots, N\}$ are mutually independent and correspond to two tandems of sizes $i-3$ and $N-i-2$, respectively. Therefore, the probability of success

$$P_{i,i+1} = \frac{Z_{i-3}Z_{N-i-2}}{Z_N} \approx \frac{1+\rho t_0+\rho t_0^2}{(3\rho t_0^2+1)} t_0^4 \tag{24}$$

Assuming that a node is equally likely to talk to any two of its neighbors, the link throughput $S_{i,i+1}$ is given by

$$S_{i,i+1} = \frac{\rho}{2} P_{i,i+1} \approx \frac{\rho}{2} \frac{1+\rho t_0+\rho t_0^2}{(3\rho t_0^2+1)} t_0^4 \tag{25}$$

The nodal throughput is

$$S_i = 2S_{i,i+1} = \rho \frac{1+\rho t_0+\rho t_0^2}{(3\rho t_0^2+1)} t_0^4 \tag{26}$$

To calculate the capacity, note that once a node is permitted to start a transmission, the success of transmission is guaranteed. Therefore, the capacity is achieved at $\rho \mapsto \infty$. For large $\rho$ the pole can be approximated by $t_0 \approx \rho^{-\frac{1}{3}}$. With this approximation

$$S_i = \frac{\rho(1+\rho^{\frac{2}{3}}+\rho^{\frac{1}{3}})}{3\rho^{\frac{1}{3}}+1} \rho^{-\frac{4}{3}} \mapsto \frac{1}{3} \tag{27}$$

The throughput per node is $\frac{1}{3}$. This is what one expects: as $\rho \mapsto \infty$ the tandem will be densely packed with transmissions. One would expect every 3-rd node to be active.

It is interesting to compare the above results to the simulation studies on ring networks reported in [Toba83]. The link throughput for a ring under C-BTMA exhibits a quasi-periodicity of period 3: all rings with a number of nodes which is a multiple of 3 have a little higher throughput than those which are not multiples of 3. The difference decreases as the number of nodes increases. This can be explained using the canonical approximation. First, note that tandem and ring exhibit similar behavior, especially for large $N$. The grand partition function has three roots - one is $t_0$ (real, positive and smallest in magnitude) and two are complex conjugates. If one considers the exact expression of the partition function, it is dominated by the smallest root. However, for small $N$, the contributions from the complex roots are not negligible. These contributions are the largest when $N$ is a multiple of 3. As $N$ increases, the contributions from these complex roots become smaller as the partition function is increasingly dominated by $t_0$. For large $N$, the partition function is insensitive to the divisibility of $N$ by 3. The simulation results were reported for relatively small size ($N < 20$) rings.

Figure 1 gives the the nodal throughput and compares it to the one using CSMA with perfect capture. For $\rho < 0.43$, CSMA outperforms C-BTMA. This says that for lighter loads, there is no need to be overly restrictive ("conservative"). For heavier loads when there is a lot of interference, being restrictive helps.

### 4.3. LINEAR ARRAY: CSMA

Consider a packet radio network of $N$ nodes placed on a linear array of degree $2d$ - nodes can transmit up to $d$ nodes in either direction. To calculate the partition function $Z_N$ one can derive a recursive relation for $\alpha_N^i$.

Suppose one more node is added to the network. Let's examine a configuration involving $i$ transmissions. There are clearly two cases to consider:

Case 1: the $N$+1-st radio is not transmitting. There are $\alpha_N^i$ such configurat

Case 2: the $N+1$-st radio is involved in a transmission. There are $\alpha_{N-d}^{i-1}$ such configurations.

Therefore,

$$\begin{aligned} \alpha^i_{N+1} &= \alpha^{i-1}_{N-d} + \alpha^i_N & N \geq d \\ \alpha^i_N &= 1 & N \leq d \end{aligned} \tag{28}$$

The above relation implies the following

$$\begin{aligned} Z_{N+1} &= Z_N + \rho Z_{N-d} & \text{for } N \geq d \\ Z_N &= 1 + \rho N & \text{for } 0 \leq N < d \end{aligned} \tag{29}$$

It follows then

$$Z_G(t) - \sum_{k=0}^{d} Z_k t^k = t\left[Z_G(t) - \sum_{i=0}^{d-1} Z_i t^i\right] + \rho t^{d+1} Z_G(t)$$

which after some algebraic manipulations reduces to

$$Z_G(t) = \frac{t - 1 + \rho t^{d+1} - \rho t}{(t-1)(1 - t - \rho t^{d+1})} \tag{30}$$

The residue of the grand partition function at its smallest pole $t_0$ (see Appendix 1 for the existence and uniqueness of this pole)

$$Res[Z_G(t_0)] = -\frac{-\rho t_0^2}{(1-t_0)[1 + d(1-t_0)]} \tag{31}$$

Applying the canonical approximation, one obtains the following expression for the partition function:

$$Z_N \approx -\frac{Res[Z_G(t_0)]}{t_0^{N+1}} = \frac{\rho t_0^2}{(1-t_0)[1+d(1-t_0)]t_0^{N+1}} \tag{32}$$

To calculate the throughput, consider a typical node $i$ and let $S_{i,i+k}$ $(0 < k \leq d)$ be the throughput of a link connecting nodes $i$ and $i+k$ which are $k$ hops apart. Since node $i$ can communicate with up to $d$ successive nodes in either direction, under the perfect capture assumption, the transmission to node $i+k$ $(0 < k \leq d)$ will be successful if the set of nodes $A_{i,i+k} = \{i-d, i-d+1, \ldots, i-1, i, i+1, \ldots, i+k+d\}$ are silent at the initiation of that transmission. The probability of success is therefore

$$P_{i,i+k} = P(A_{i,i+k} \text{ idle}) = \frac{Z_{N \setminus A_{i,i+k}}}{Z_N} =$$

$$= \frac{Z_{i-d-1} Z_{N-i-k-d}}{Z_N} \approx -Res[Z_G(t_0)] t_0^{2d+k} \tag{33}$$

Let us consider the case when the traffic is equally distributed among $2d$ outgoing links from node $i$, that is $\rho_{i,i+k} = \rho/2d$. In such a case, the link throughput

$$S_{i,i+k} = \frac{\rho}{2d} P_{i,i+k} \approx \frac{\rho^2 t_0^{2d+k+2}}{2d(1-t_0)[1+d(1-t_0)]} \tag{34}$$

The nodal throughput of $i$,

$$S = 2\sum_{k=1}^{d} S_{i,i+k} \approx \frac{\rho^2 t_0^{2d+3}}{d(1-t_0)[1+d(1-t_0)]} \frac{1-t_0^d}{1-t_0} =$$

$$= \frac{t_0(1-t_0^d)}{d[1+d(1-t_0)]} \tag{35}$$

Figure 2 gives the curves of the nodal throughput as a function of load for $d = 3$, $d = 5$ and $d = 10$. The corresponding capacities are $S_i = 0.0826$ at $\rho = 0.735$, $S_i = 0.0544$ at $\rho = 0.525$ and $S_i = 0.0293$ at $\rho = 0.31$.

Let us calculate the capacity for large $d$. From equation (35) one differentiates the nodal throughput $S_i$ with respect to $t_0$ and solves the equation $S'(t_0) = 0$.

One would obtain

$$d^2 t_0^{d+1} - (d+1)^2 t_0^d + d + 1 = 0 \tag{36}$$

For large $d$, the pole $t_0$ is very close to 1. Therefore, writing $t_0 = 1 - \alpha$ and using the approximation $(1-\alpha)^d \approx 1 - d\alpha$ one gets that the

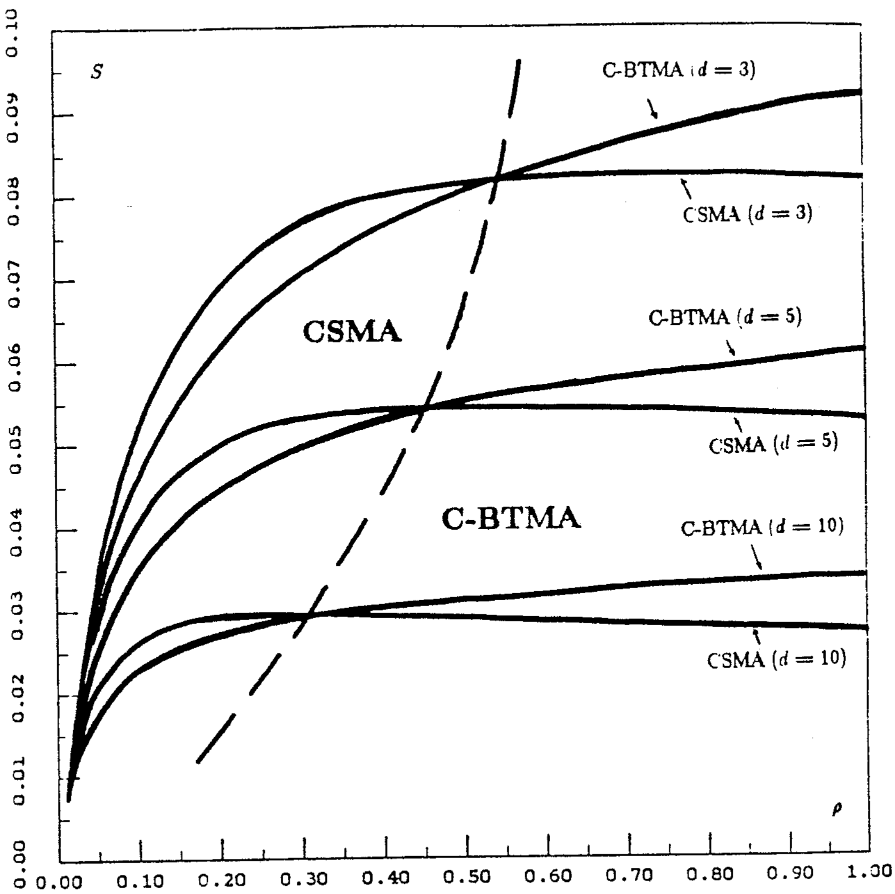


**Figure 2:** Nodal Throughput: CSMA and C-BTMA (d=3,5,10)

capacity is achieved at

$$t_0 \approx 1 - \frac{1}{d+1} \tag{37}$$

From equation (35) one gets that the capacity is

$$S_i \approx \frac{\frac{d}{d+1}\left[1 - \left(1 - \frac{1}{d+1}\right)^d\right]}{d\left(1 + \frac{d}{d+1}\right)} \approx \frac{1 - \frac{1}{e}}{2d} = \frac{0.318}{d} \tag{38}$$

The corresponding load

$$\rho = \frac{1 - t_0}{t_0^{d+1}} \approx \frac{e}{d} \tag{39}$$

## 4.4. LINEAR ARRAY: C-BTMA

Consider the same linear array of packet radio nodes but now operating under the C-BTMA scheme. As before, one can derive a recurrence relation for the partition function by adding one more node and considering the corresponding two cases. Because of the C-BTMA the recurrence relation becomes $\alpha_{N+1}^i = \alpha_N^i + \alpha_{N-2d}^{i-1}$ for $N \geq 2d$ with initial conditions $\alpha_N^1 = N$ for $N \leq 2d$

This gives

$$\begin{aligned} Z_{N+1} &= Z_N + \rho Z_{N-2d} && \text{for } N \geq 2d \\ Z_N &= 1 + \rho N && \text{for } 0 \leq N < 2d \end{aligned} \tag{40}$$

The grand partition function $Z_G(t)$ can be shown to satisfy

$$Z_G(t) = \frac{t - 1 + \rho t^{2d+1} - \rho t}{(t-1)(1 - t - \rho t^{2d+1})} \tag{41}$$

Let $t_0$ be the smallest positive pole. (See Appendix for the existence and uniqueness of this pole.) The residue is then

$$Res[Z_G(t_0)] = -\frac{t_0 - 1 + \rho t_0^{2d+1} - \rho t_0}{(t_0 - 1)(1 + \rho(2d+1)t_0^{2d})} =$$

$$= -\frac{\rho t_0^2}{(1-t_0)[1+2d(1-t_0)]} \tag{42}$$

The partition function is given by

$$Z_N \approx -\frac{Res[Z_G(t_0)]}{t_0^{N+1}} = \frac{\rho t_0^2}{(1-t_0)[1+2d(1-t_0)]t_0^{N+1}} \tag{43}$$

As in the previous case, define $S_{i,i+k}$ $(0 < k \leq d)$ to be the link throughput between nodes $i$ and $i+k$ that are $k$ hops apart. Because of the C-BTMA, the node $i$ it will be successful at every transmission to $i+k$. It can initiate a transmission if the set of nodes $A = \{i-2d, i-2d+1, \ldots, i+2d\}$ is silent. The probability of that is

$$P_{i,i+k} = \frac{Z_{i-2d-1} Z_{N-i-2d}}{Z_N} \approx Res[Z_G(t_0)]t_0^{4d} =$$

$$= \frac{\rho t_0^{4d+2}}{(1-t_0)[1+2d(1-t_0)]} \tag{44}$$

Assuming that node $i$ is equally likely to talk to any of its $2d$ neighbors, the link throughput is given by

$$S_{i,i+k} = \frac{\rho}{2d} P_{i,i+k} \approx \frac{\rho^2 t_0^{4d+2}}{2d(1-t_0)[1+2d(1-t_0)]} \tag{45}$$

The nodal throughput

$$S_i = 2\sum_{i=1}^{d} S_{i,i+k} \approx \frac{\rho^2 t_0^{4d+2}}{(1-t_0)[1+2d(1-t_0)]} \tag{46}$$

Since every transmission results in success, the capacity is achieved at $\rho \mapsto \infty$. For large $\rho$ the pole can be approximated as follows

$$t_0 \approx \rho^{-\frac{1}{2d+1}} \tag{47}$$

The asymptotic nodal throughput is easily seen to be $S_i = 1/(2d+1)$. This is what one would intuitively expect - for very large $\rho$ the linear array will be packed with transmissions. Since every $i$-to-$j$ transmissions blocks $2d+1$ nodes and the system is densely packed, one would expect that the throughput per node is $1/(2d+1)$.

It is interesting to compare the capacity $S_i^{max}$ for CSMA with perfect capture and C-BTMA for large $d$. For CSMA we obtained that

$$S_i^{max} \approx \frac{0.318}{d} \tag{48}$$

For C-BTMA

$$S_i^{max} \approx \frac{0.5}{d} \tag{49}$$

This says that for the same (large) $d$ the capacity under CSMA i: 36% worse than under C-BTMA.

Figure 2 gives the nodal throughput and compares it with CSM/ with perfect capture for $d = 3, d = 5$ and $d = 10$. Just as in the case c $d = 1$, for lighter loads CSMA outperforms C-BTMA. However, we not that as the level of interference increases ($d$ increases), the range of th loads for which CSMA is better decreases. The reason is that one shoul be more and more restrictive as the interference increases.

## 5. CONCLUSION

This paper presented an application of the new method of canor cal approximation to analyze CSMA and C-BTMA protocols with perfe capture for a number of network topologies. The method allows to analy a number of network topologies with relative ease. Further work will exte the application of the new method to study other multiple access protoc and network topologies.

# APPENDIX 1

**Theorem 1.** Let $t_0$ be the smallest pole of the grand partition function. Then for large $N$ one has ([Henr77])

$$Z_N \approx \frac{-Res[Z_G(t_0)]}{t_0^{N+1}}$$

**Proof.** Let $t_0, t_1, ...t_k$ be the singularities of $Z_G(t)$. Assume that there are a finite number of poles and all poles are of order 1. For more general case, the proof proceeds along similar lines and is given in [Pins85].

$$0 < |t_0| < |t_1| < ... < |t_k|$$

Let $C$ be the circle around the origin excluding all the poles and let $C'$ be the circle around the origin surrounding all the poles of $Z_G(t)$.

Then by residue theorem ([Alfh66])

$$Z_N = \frac{1}{2\pi i}\oint_C \frac{Z_G(t)}{t^{N+1}}dt = -\sum_{i=0}^{k} \frac{Res[Z_G(t_i)]}{t_i^{N+1}} + \frac{1}{2\pi i}\oint_{C'} \frac{Z_G(t)}{t^{N+1}}dt =$$

$$= -\frac{Res[Z_G(t_0)]}{t_0^{N+1}}\left[1 + \sum_{i=1}^{k} \frac{Res[Z_G(t_i)]}{Res[Z_G(t_0)]}\left(\frac{t_0}{t_i}\right)^{N+1}\right] + \frac{1}{2\pi i}\oint_{C'} \frac{Z_G(t)}{t^{N+1}}dt$$

But since

$$\left|\frac{1}{2\pi i}\oint_{C'} \frac{Z_G(t)}{t^{N+1}}dt\right| \leq \frac{\max_{t\in C'}[Z_G(t)]}{|t|^N}$$

and $|\frac{t_0}{t_i}| < 1$ one obtains that for large $N$

$$Z_N = \frac{-Res[Z_G(t_0)]}{t_0^{N+1}}[1 + \mathrm{E}]$$

With the (relative) error of the approximation

$$\mathrm{E} \leq \left[\sum_{i=1}^{k} \frac{Res[Z_G(t_i)]}{Res[Z_G(t_0)]}\left(\frac{t_0}{t_i}\right)^{N+1}\right] + \frac{\max_{t\in C'}[Z_G(t)]}{Res[Z_G(t_0)]}\left(\frac{t_0}{|t|}\right)^{N+1} \mapsto 0$$

Therefore, for large $N$

$$Z_N \approx \frac{-Res[Z_G(t_0)]}{t_0^{N+1}} \tag{50}$$

The partition function is asymptotically determined by the smallest pole of its grand partition function. ∎

To use the above approximation, one must be able to compute residues fast. The calculation of residues becomes very easy with the help of the following theorem ([Alhf66]):

**Theorem 2.** Suppose $f(t)$ has a pole of order $m$ at $t = t_0$ and put $g(t) = (t - t_0)f(t)$. Then the residue of $f(t)$ at point $t_0$ is given by

$$Res[f(t_0)] = \frac{1}{(m-1)!} g^{(m-1)}(t_0) \tag{51}$$

In particular, if $Z_G(t)$ is of the form

$$Z_G(t) = \frac{F(t)}{1 - t - \rho t^{n+1}}$$

and $t_0$ is a simple pole of $Z_G(t)$ then

$$Res[Z_G(t_0)] = -\frac{F(t_0)}{(n+1)\rho t_0^n + 1} \tag{52}$$

∎

**Theorem 3.** The function $f(t) = 1 - t - \rho t^n = 0$ has only one positive root $t_0$. This root is of order 1 and is the smallest in magnitude among all other roots.

**Proof.** Since $f(0) = 1 > 0$ and $f(1) = -\rho < 0$ the function has a root $t_0$ satisfying $0 < t_0 < 1$. Moreover, since $f'(t) < 0$ for $t > 0$ it follows that $f(t)$ is a decreasing function for positive $t$ and hence, the root is unique and of order 1.

For any other root $t_1$ one can write $t_1 = R\cos\theta + iR\sin\theta$ where $\theta \neq 0$. Since $t_1$ satisfies $1 - t_1 - \rho t_1^n = 0$ one obtains

$$g(R,\theta) = 1 - R\cos\theta - \rho R^n \cos n\theta = 0 \tag{53}$$

But $g(R,\theta) > 1 - R - \rho R^n = f(R)$. The function $f(R)$ has only one positive real root $R = t_0$. Moreover, the function $f(R)$ is decreasing for larger $R$. Therefore, $g(R,\theta)$ is bounded below by a decreasing function intersecting the $X$-axis at $t_0$. But this implies $t_1 > t_0$. ∎

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